Measure Theory
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Module No # 12
Lecture No # 58
Borel and Radon measures of LCH spaces

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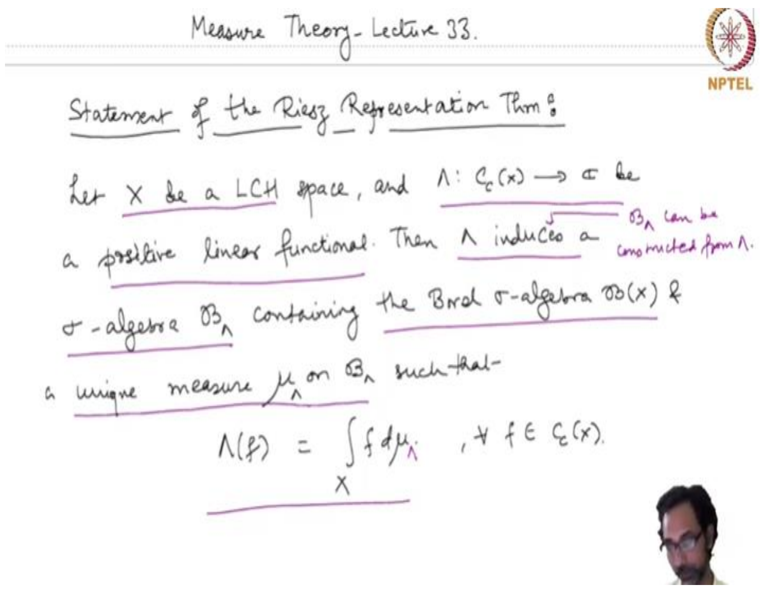

So let us look at this statement of the Riesz representation theorem it states that if X is a locally compact Hausdorff space. And lambda is positively near functional on the space of continuous compactly supported functions on X. Then lambda induces a sigma algebra B lambda meaning that here induces means that B lambda can be constructed from lambda. So this is the statement saying that the lambda induces the sigma algebra B lambda meaning that B lambda can be constructed just from the given lambda.

And this sigma algebra B lambda contains the Borel sigma algebra B x so remember that the Borel sigma algebra was the collection of all Borel subsets of X which is the smallest sigma algebra generated by the open subsets of X. And so the sigma algebra B lambda that lambda induces it contains all the Borel sets and a unique measure mu lambda defined on B lambda such that the positively linear functional lambda applies to f is nothing but the integral of, f with respect to this measure mu lambda.

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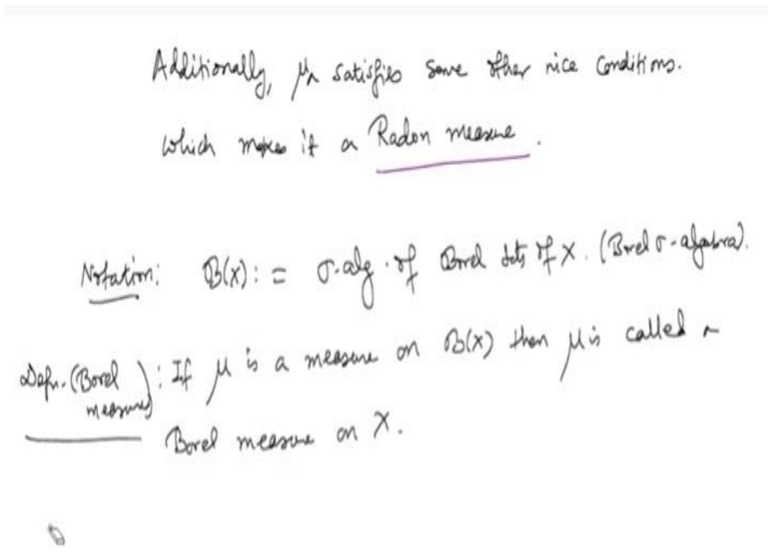

Additionally mu lambda satisfies some extra nice conditions some other nice conditions properties of regularity which we call regularity of the measure which makes it a, what is called a Radon measure. So this mu lambda that is induced by the positively linear functional lambda is the so called Radon measure. So before we come to the proof of the Riesz representation theorem let us look at the definition of Radon measure and see some nice properties that Radon measures enjoy on locally compact Hausdorff spaces.

So first let me fix some notation so B x is the sigma algebra of Borel sets of x so this is called the Borel sigma algebra. And if mu is a measure on this sigma algebra of Borel subsets then mu is called a Borel measure on x. So this is the definition of a Borel measure so, now if so let me go to another page.

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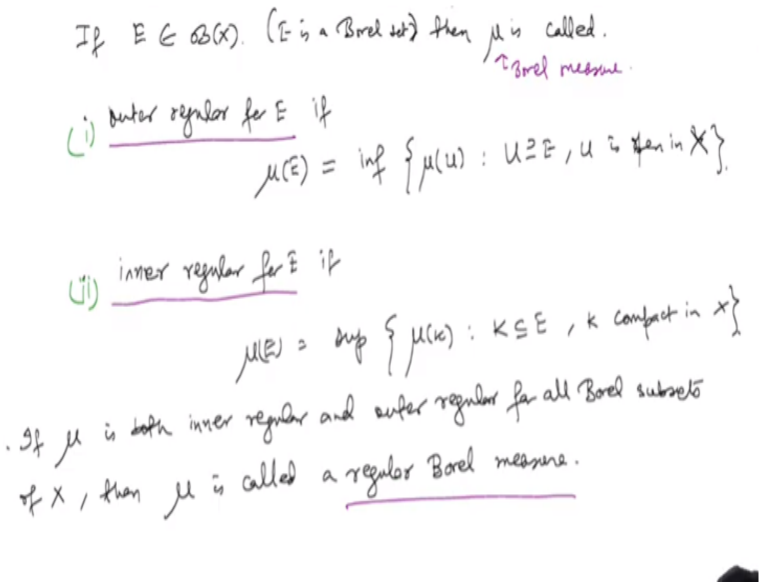

So if E belongs to B of x meaning that E is a Borel set then mu is called outer regular for E if mu of E so here mu is Borel measure. So this is a Borel measure and if mu of E is the infimum of mu of u such that u contains E and u is open in X. So if this equality holds then mu is called outer regular for E similarly mu is called inner regular for E if mu of E is now the supremum over mu k.

Such that k is the subset of E which is compact so in this case mu is called inner regular for E if mu is both inner regular and outer regular for all Borel subsets in x of x. Then mu is called a regular a Borel measure this is a regular Borel measure so if both this conditions so let me name them 1 and 2. So if both 1 and 2 are satisfied so it is both outer regular and inner regular for all Borel sets then mu is called a regular Borel measure.

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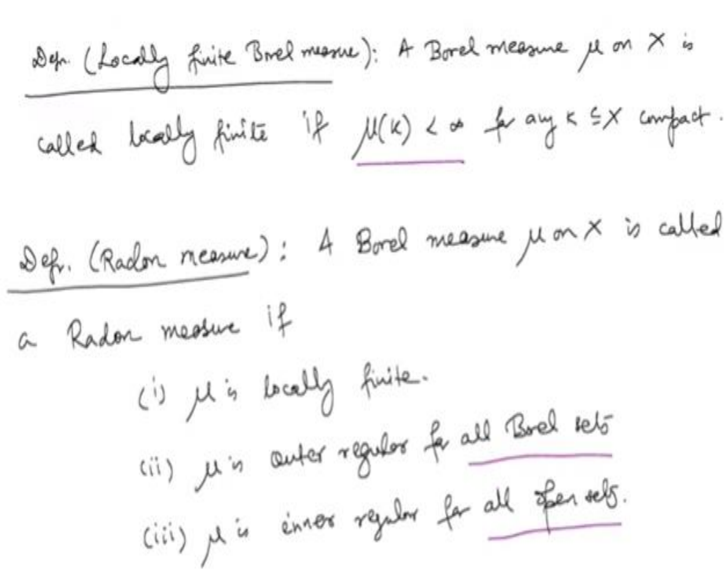

Now another definition this is about locally finite Borel measure so a Borel measure mu on x is called locally finite. If measure of all compact sets k is finite for any k compact we have that the measure of k is finite. So finally we come to the definition of the Radon measure and this is a measure a Borel measure mu on x is called a Radon measure. If mu is locally finite mu is outer regular for all Borel sets and mu is inner regular for all open sets.

So here we allow outer regularity for all Borel sets but inner regularity for only open subsets in x. So a locally finite measure which is outer regular for all Borel sets and inner regular for all open sets is called a Radon measure.

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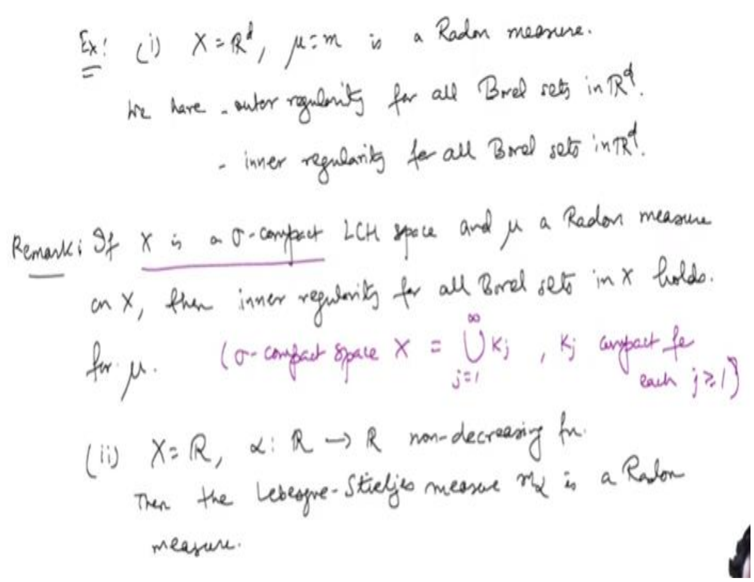

So what are the examples of Radon measures that we can construct so of course on Rd the Lebesgue measure is a Radon measure. Because of course all compact sets have finite Lebesgue measure because they are bounded so they have finite Lebesgue measure. And we have already seen that for all Lebesgue measureable sets we have outer regularity as well as inner regularity. So in particular we have outer regularity for all Borel sets and we have inner regularity also for all Borel sets in Rd.

So not just for open sets inner regularity holds for all Borel sets and in fact it we have seen that it also holds for all Lebesgue measureable sets. But for the purposes of seeing viewing it as a Radon measure it is enough to consider outer regularity for all Borel sets and irregularity for all open sets. And here there is a (()) (11:20) that if your locally compact Hausdorff space so if so let me put this as a remark that if x is a sigma compact space locally compact Hausdorff space.

And mu a Radon measure on X then inner regularity for all Borel sets in X holds for mu. So a Radon measure by our definition as allowed to only have inner regularity for all open sets but in the case of a sigma compact space. So recall that sigma compact space x is countable union it can be expressed as a countable union of compact space kj compact for each j. So when you have a sigma compact space then inner regularity for all open sets implies inner regularity for all Borel sets.

So now we have seen that our Lebesgue measure on Rd is a Radon measure in fact if you consider alpha. So on X if you take X to be the real line alpha is a non-decreasing map non-decreasing function on R. Then the Lebesgue Stieljis measure that we have constructed before m alpha is a Radon measure. So all the nice examples of measures that, we have seen until now they are Radon measure.

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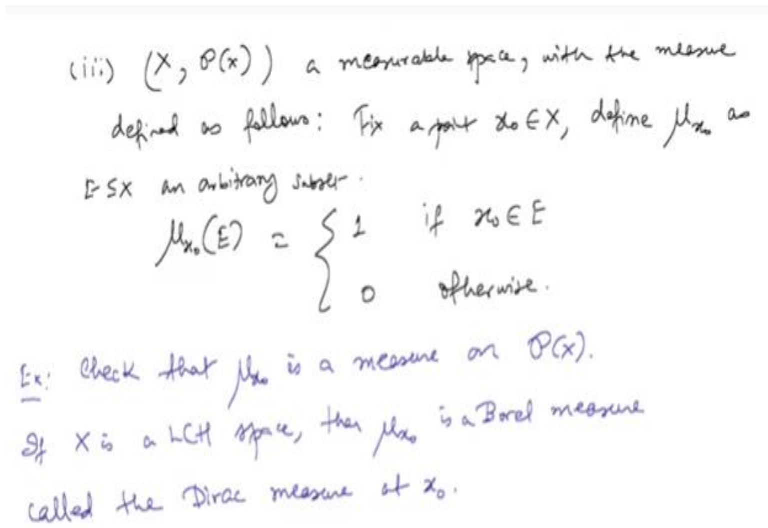

Now let us look at another example for abstract spaces X an abstract space let us take an X and p x as a measureable space with the following measure with the measure defined as follows. Let us fix a point X naught in X then we define the measure mu x naught as follows mu X naught of any subset of E of X. So E an arbitrary subset then you can define mu X naught of E equals 1 if X naught is in E and 0 otherwise.

So this kind of so one can check actually so let me put this as an exercise is to check that mu X naught is a measure on defined on the sigma algebra of all subsets of X. So in particular it is f x is a topological space a locally compact Hausdorff space then it will contain all the Borel subsets of x and it will be a Borel measure. And now if x is a locally compact Hausdorff space so this by this remark mu X naught is a Borel measure this is called the Dirac measure at X naught.

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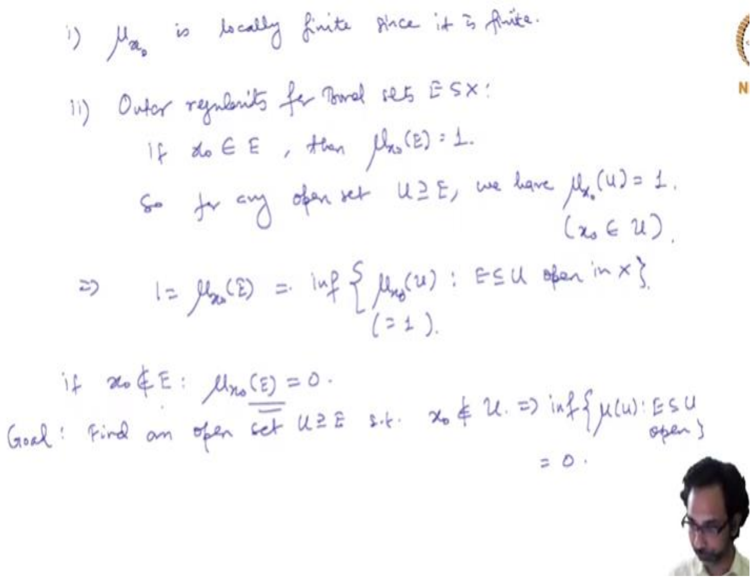

So now let us check whether this is a Radon measure so first is that mu X naught is locally finite since it is finite so it is finite measure. So therefore it is locally finite because it takes only values 0 and 1 secondly for outer regularity for Borel subsets we have the following. So if x naught belongs to E then mu x naught of E is 1 so for any open set u containing E we have mu x naught of u is also equal to 1 because x naught belongs to u.

And this implies that 1 which is equal to mu x naught E is equal to the infimum of all mu x naught u. Such that E is the subset of u which is open in x because all of these are equal to this is equal to 1. So this whole set is the just 1 number which is 1 so the infimum of just a singleton set is the value of that set. So we get 1 so outer regularity for Borel sets holds in this case on the other hand if x naught is does not belong to E we proceed as follows.

So of course then mu x naught E equal to 0 and now we have to find an open set containing E which does not contain x naught. So goal our goal is to find an open set u containing E such that x naught does not belong to u. So this would imply that the infimum of mu u such that E is the subset of u and u is open this is equal to 0 and this would imply that it is it also satisfies outer regularity when x naught does not belong to E.

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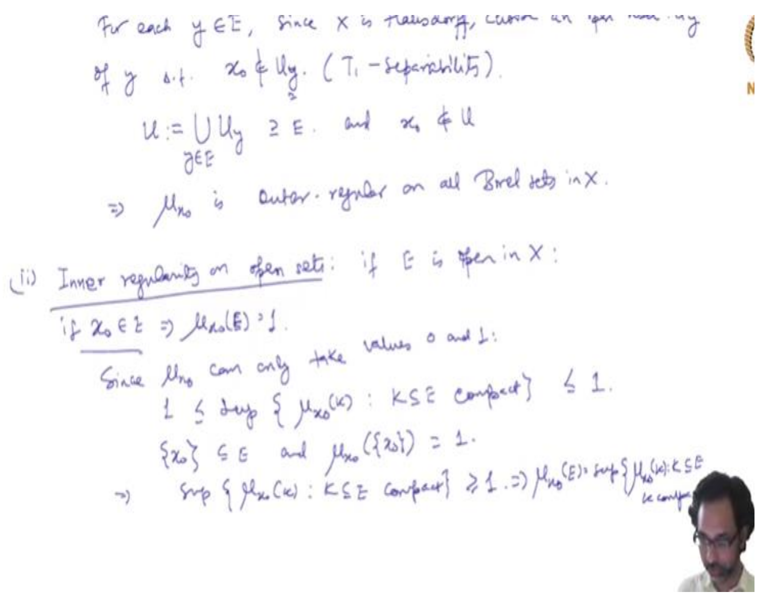

So let us say how this can be done so for each y in E because x is Hausdorff since x is Hausdorff we choose an open set an neighborhood u y of y such that x naught does not belong to u y. So in fact this is does not even require the full Hausdorff condition is just t1 so this is the t1 separability axiom. Now we can write we can take the union over y in E of this open sets u y. So this is an open sets u and this contains E because the union is over all points in E and x naught does not belong to u because x naught does not belong to any of the u y's.

So x naught does not belong to u. So this means that our mu x naught is outer regular on all Borel sets in X. Now for inner regularity on open sets so if E is open in x now we can again we can divide it 2 cases. If x naught belongs to e then again mu x naught of E equals 1. And now we have to take the supremum of all compact subsets of E and see whether the measures of the supremum of those measures of compact subsets inside E also gives you 1.

Now since mu x naught can only take values 0 and 1 then this implies that the supremum of mu x naught k such that k sitting inside E is compact this is of course less than or equal to 1. On the other hand they have the compact set the singleton set the x naught this is a compact set this sits inside E and mu x naught of this compact set x naught this is equal to 1. So this implies that the supremum is also greater than or equal to 1 which proves that this is in fact.

So equal to 1 so this is implies that u x naught E is equal to the supremum of mu x naught k such that k is in E and k compact. So this is the case when x naught belongs to E.

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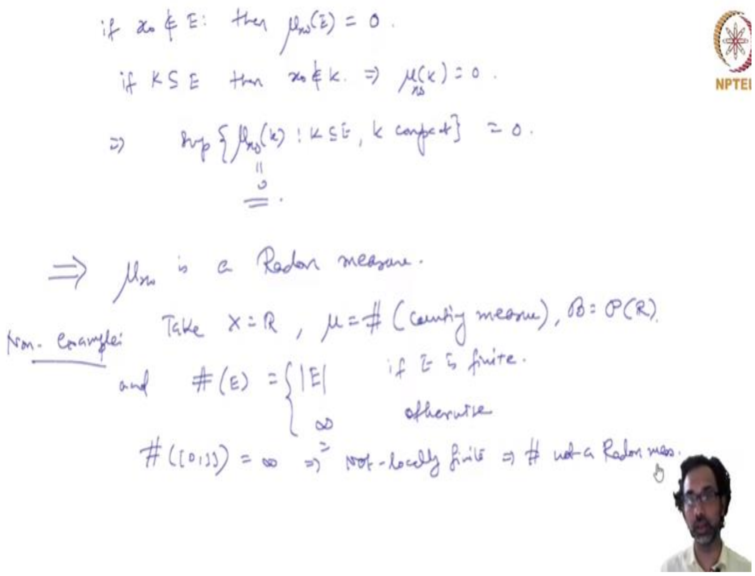

On the other hand if x naught does not belong to E then of course mu x naught E is equal to 0. But now it is obvious that if k is any subset of E then x naught does not belong to k, and this implies that mu naught k equal to 0, which means that the supremum of mu x naught k for if you only consider compact sets then also this is going to 0 all of these are 0. So we see that it is both inner and outer regular and it is locally finite so this means that mu x naught is a Radon measure.

Now let us see a non-example so it is an example of a measure which is not a Radon measure so we can take x as Rd even just R. And mu to be the counting measure so what is the counting measure so this is defined on all subsets of R and this counting measure of a set E is just the cardinality of E. So if E is finite and infinity otherwise so if E not finite it is infinite and so the cardinality we set it to be infinity.

So of course if you have a compact set so let us say the set 0, 1 the interval 0, 1 this is a compact subset of r. But this gives you an value infinity so this means that it is not locally finite implies that this is not a Radon measure. So one can even have examples where the measure is finite but not either inner regularity fails or outer regularity fails but I will not go into this examples. You can look at chapter 7 in Folland's books for those kinds of examples where your measure is finite but it is not Radon meaning that either outer regularity or inner regularity fails to hold okay.