Measure Theory
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Module No # 12
Lecture No # 57
Basics on Locally compact Hausdorff spaces

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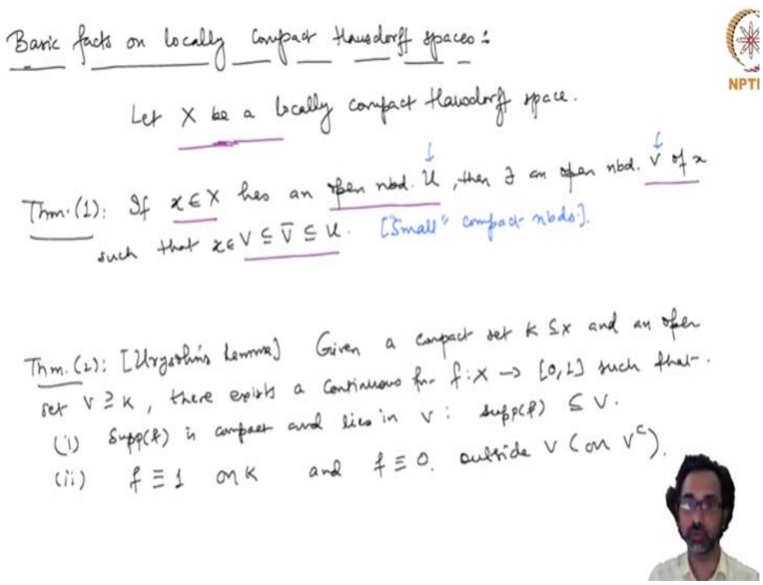

So let us consider some basic facts on locally compact Hausdorff spaces of course we need Hausdorffness to have some nice analytic properties their existence of unique limits and so on. So we will consider those topological spaces which enjoy both local compactness as well as Hausdorff property. So if you start with a locally compact Hausdorff space X then the first theorem space that if X is a point in X then it as an open neighborhood u then their exist in open neighborhood B V of x such that v the closure of V sits inside u.

So this is about small compact neighborhood existence of small compact neighborhoods meaning that no matter how small this open neighborhood u is there is something smaller v is something smaller which contains whose closure is contained in u. So this is the first used nice theorem that we shall need and the second one is a very important result in topology and this is called Urysohn’s lemma even it is called lemma.

I will state is as theorem then it is quite important and this Urysohn's lemma states that given a compact set k in x and an open set V which contains k. So open set V containing k their exist a continuous function f from x to C such that the first property is that the support of f is compact and lies in v meaning that support of, f is subset of V. And secondly at f is identically equal to 1 on K and is f is 0 outside V meaning on V complement is 0.

So Urysohn's lemma guarantee's the existence of such a function on locally compact Hausdorff spaces actually here we can actually take the range to be 0, 1 rather than the whole complex numbers because f only takes value between 0 and 1. So we will use Urysohn's Lemma to create what are called partitions of unity.

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Thm. [Partition of unity] : Let X be a LCH space, $V_1, V_2, \dots, V_n$ are open in X and K is compact in X s.t.

$$K \subseteq \bigcup_{i=1}^{n} V_i$$

Then there exist fns. $\varphi_i \prec V_i$ such that for all $x \in K$, we have

$\varphi_i \in C_c(X)$, $0 \le \varphi(x) \le 1$ and $\text{supp}(\varphi_i) \subseteq V_i$.

$$\sum_{i=1}^{n} \varphi_i(x) = 1$$

"partition" of unity.

$\{\varphi_i\}_{i=1}^{n}$ is called a partition of unity subordinate to $\{V_i\}_{i=1}^{n}$.

So before I go to the partition of the unity result let us fix some notation so we will see that we will write that k let me call it less than f. So this will mean that k is compact f is a continuous function with compact support x such that it lies between the range of f lies between 0 and 1 and then finally that f is identically equal to 1 on k. S this is the notation in this case a second notation that we will use is f less than v and this will mean that v is an open set f is still continuous function with compact support with values between 0 and 1.

And that support of, f is contained inside v so this means that f is identically equals to 0 on v complement. So in terms of the notation Urysohn's lemma with this notation states that given k compact sitting inside v open this is compact and this is open, their exist a continuous function with compact support such that k is the sub set of, f less than f is less than v. So I am putting together these 2 parts of the notation and this means that f as range between 0 and 1 and f is 1 on k and f is 0 on k and f is 0 on the complement of v.

Note that this is also same as saying that this also means that we have an inequality that we have point wise inequality with the indicative function of the compact set k is less than or equal to f which is less than or equal to the indicative function of v. So we are trying to sandwich a continuous function in between the indicative function of a compact set and the indicative function of an open set. So let us see the statement of partition of unity theorem.

It says that on a locally compact Hausdorff space x if we have a finite collection of open sets v1, v2, v n these are all open in x and if we have a compact set k is compact which is contained in the union of these V i's. Then their exist functions Phi i less than V i remember that these means that support of first that Phi i is a continuous function with compact support with values between 0 and 1 and support of Phi i is a sub set of V i.

So their exist for each i for function continuous function of compact support Phi i whose support lies inside Vi and such that for any x in this set compact set k we have that the sum of Phi i from 1 to n evaluated at x sums up to 1. And this is called a partition of unity because you are waking up this function which is identically 1 on k in to parts which as support inside compact inside V i. So this collection Phi i, i = 1 to, n is called a partition of unity subordinate to the collection V i, i = 1 to, n.

So let us try to see how to prove this partition of unity existence of this partition of unity using the Urysohn's Lemma.

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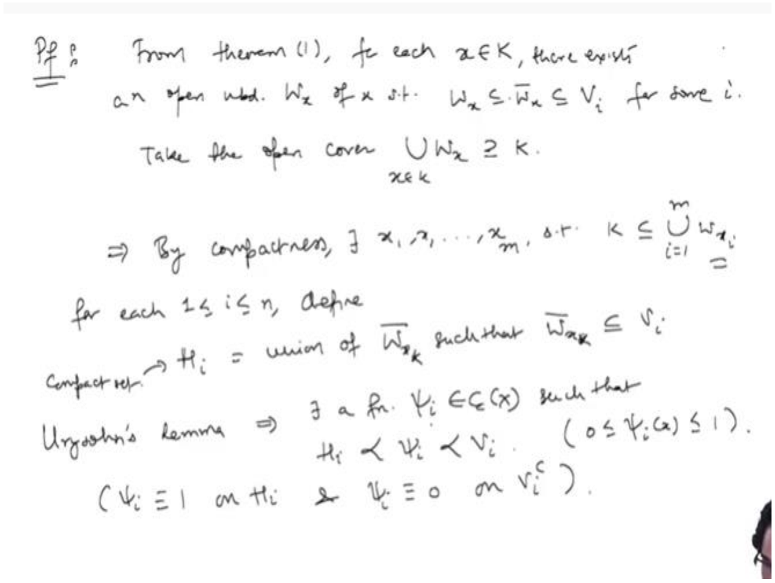

So to prove this we will use the first theorem that we saw about small existence of small compact above. So from theorem 1 above for each x in the compact set k there exist a neighborhood and open neighborhood let us call it W x of x such that the closure of W x lies in some V i for sum i. Now we can take the open cover the unions of these W x the x in k and this contains k. So by compactness there exist x1, x2 finitely many points x m say such that k is a subset of the finite union i = 1 to m W x i.

And so for each i in 1 to n define H i to be the union of the compact sets W x i bar. Such that these W x let us write k here W x k bar such that W x k bar lies in V i so once you fix i we only choose those W x i's whose closure is a subset of V i so this union is a compact set sitting inside Vi. So now Urysohn's lemma implies that there exist a function C i continuous function with compact support such that H i less than C i less than V i.

So this means that C i is identically 1 on H i and vanishes outside V i C i is identically 1 on Hi and C i is 0 on V i complement. And of course we have 0 less than equal to C i less than equal to 1 for all x so now we will define our Phi i's the require our partition of unity using these C i's.

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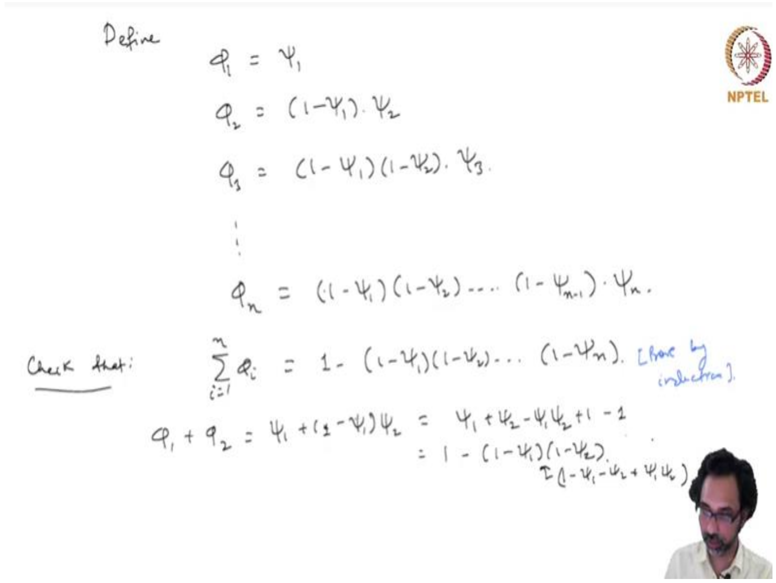

So define Phi 1 to be C1 Phi 2 to be 1 – C1 times C2 Phi 3 to be 1 – C1 1 – C2 times C3 and so on. We can define for each i from 1 to n Phi i the last one will be the product 1 – C1, 1 – C2 and so on up to 1 – C n – 1 and multiplied by C n. So now check the following formula that the summation of Phi i, i = 1 to, n is nothing but 1 – C1, 1 – C2 up to 1 – C n. So for example you can prove this by induction. For example if you have just 2 Phi 1 + Phi 2 this is 1 – so first one is C1 + 1 – C1 times C2 and so this is equal to C1 + C2 – C1 C2.

And if you add and subtract 1 so let me add 1 and subtract 1 and we can collect so I will take the positive one and we can collect the other terms as 1 – C1 times 1 – C2. Because this is nothing but 1 – C1 – C2 + C1 C2 and so there is a minus sign here which takes care of the, which gives you the right formula. So we can do it for 2 and then assume for i and rather assume for n – 1 and prove it for n. So I will leave it to you as an exercise to prove this result by induction.

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Since $K \subseteq \bigcup_{i=1}^{n} H_i$, for each $x \in K$, $x \in H_i$ for some i.

and so $\psi_i(x) = 1$

$$\Rightarrow \left[(1-\psi_1)(1-\psi_2)(1-\psi_3)\cdots(1-\psi_n)\right](x) = 0.$$

$$\Rightarrow \sum_{i=1}^{n} \varphi_i(x) = 1 - \underbrace{\prod_{i=1}^{n}(1-\psi_i(x))}_{0} = 1.$$

And now since k is the subset of the union of H i's and equal to 1 to, n for each x in k x belongs to H i for sum i and therefore C i of x equals 1. So this implies that 1 – C1, 1 – C2, 1 – C3 up to 1 – C n applied it x so this function applied at x this is just the point wise multiplication this is 0 for any x. So this implies that the summation of Phi i x, i = 1 to, n which is 1 – this product let me write the product i = 1 to n 1 – C i x this is 0 so this is equal to 1. So these prove the existence of partitions of unity which we will use quite significantly in the proof of the Riesz representation theorem.