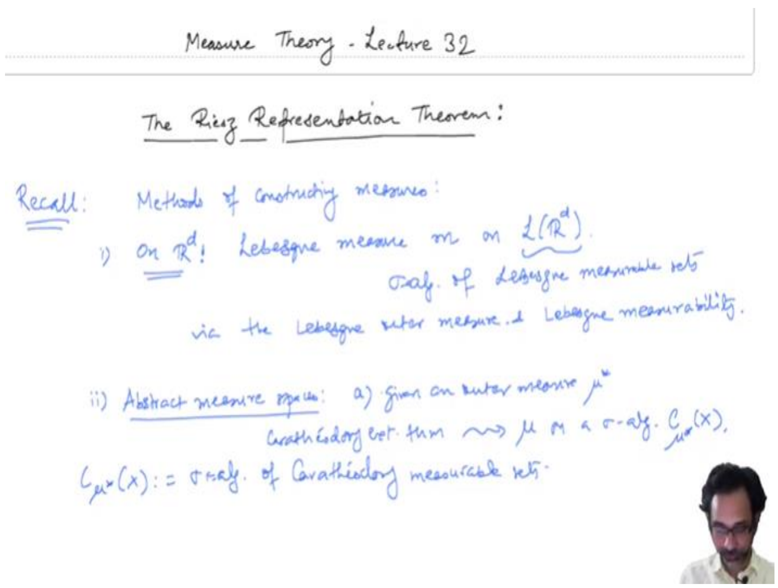
Measure Theory
Prof. Indrava Roy
Department of Mathematics
Institute of Mathematical Science

Module No # 12
Lecture No # 56
Riesz Representation theorem – Motivation

We now come to new topic and this is about the Riesz representation theorem so this Riesz representation theorem is another method to define measures on topological spaces which have some nice properties.

(Refer Slide Time: 00:38)

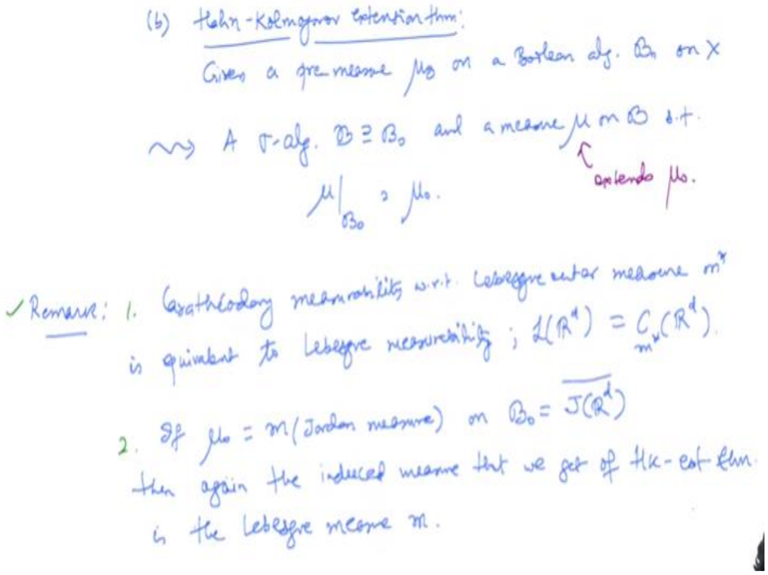


So let me recall what we have seen so far in terms of methods of constructing measures. So first we have seen on Rd we have seen the Lebesgue measure m on defined on the Lebesgue sigma algebra L of Rd. So this is the collection Lebesgue measureable sets and this forms sigma algebra so this construction was via the Lebesgue outer measure. And we defined a concept of Lebesgue measurability with respect to this Lebesgue outer measure Lebesgue measurability.

So we saw that the sigma algebra of Lebesgue measureable sets when we restrict the Lebesgue measure on the sigma algebra it has some nice properties namely it has the countable additivitiy property for disjoint collections of Lebesgue measurable sets. So this was specifically on Rd then on abstract measure spaces we saw 2 ways of defining a measure. So the first one was via given an outer measure mu star.

Then the Caratheodory extension theorem gives you a measure mu on a sigma algebra B where rather I used notation C mu star of X. Where, C mu Star of X was the sigma algebra of Caratheodory measurable sets and this precisely the once that satisfies the Caratheodory criteria.

(Refer Slide Time: 03:45)



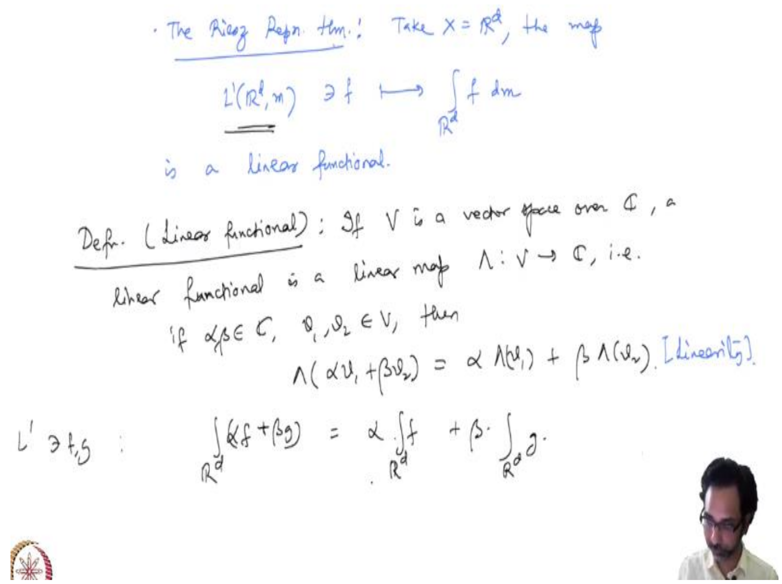
The other way was to use the Hahn Kolmogorov extension theorem and this was the input data for this was given a premeasure mu naught on a Boolean algebra B naught on X. Then the Hahn Kolmogorov extension theorem gives you a sigma algebra B which contains this Boolean algebra B naught and a measure mu on B such that mu restricted to B naught equals mu naught. So this was the extension theorem because this measure mu extends mu naught and we saw that both this methods for abstract measure spaces.

Namely the Caratheodory extension theorem and the Hahn Kolmogorov extension theorem can be applied to Rd. And when we use the Jordan measure as the premeasure on Jordan measureable Boolean algebra of Jordan measureable sets then we get back the Lebesgue measure. And similarly when we use the Lebesgue outer measure and use the Caratheodory extension theorem then we again get back the Lebesgue sigma algebra.

So Caratheodory measurability so let me put this as a remark that Caratheodory measurability which respect the Lebesgue outer measure m star is equivalent to Lebesgue measurability. So this means that the sigma algebra of Lebesgue measureable sets is the same as the sigma algebra of Caratheodory measurable sets with respect to the Lebesgue outer measure. So this we have already seen before and in terms of so this is the first remark and the second remark is that in terms of.

If we take mu naught is the Jordan measure on B naught being the Boolean algebra of Jordan measurable sets. Then again the induced measure that we get from the Hahn Kolmogorov extension theorem is the Lebesgue measure again. So in this sense these 2 approaches when applied to Rd gives you back the Lebesgue measure when you choose the Jordan measure using the Hahn Kolmogorov extension theorem. And when, you choose the Lebesgue outer measure using the Caratheodory extension theorem.

(Refer Slide Time: 08:10)

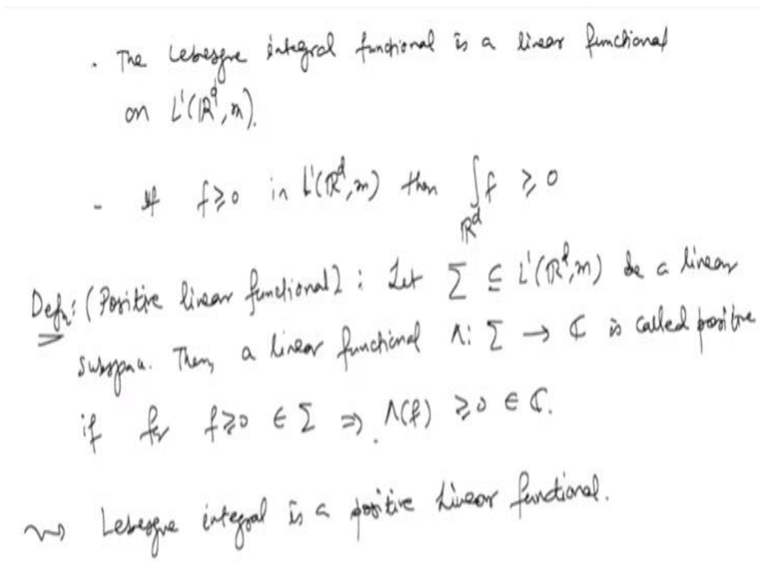


So now the Riesz representation theorem where it fits in this scheme the Riesz representation theorem fits in the following way. So if we take X to be Rd then the following map the map which takes a function in L1 of Rd with the Lebesgue to its Lebesgue integral d m this is a linear functional. So linear functional means that so let me define what is a linear functional? So if V is a vector space let us say a complex vector space and linear functional is a linear map lambda from V to C.

Meaning that if alpha beta are complex constants and v1 and v2 belong to V then we have lambda of alpha v1 + beta v2 should be equal to alpha lambda v1 + beta lambda v2. So this is the well-known linearity property and of course we have seen that the Lebesgue integration is linear in this sense because if you take integral alpha f + beta g for 2 functions f g in L1. We have that then Lebesgue integral of alpha + beta g is equal to alpha times the Lebesgue integral of f + beta times the Lebesgue integral of g.

So this is an example of a so the Lebesgue integral is an example of linear functional over L1 of Rd.

(Refer Slide Time: 11:06)



- The Lebesgue integral functional is a linear functional on $L^1(\mathbb{R}^d, m)$.
- If $f \geq 0$ in $L^1(\mathbb{R}^d, m)$ then $\int_{\mathbb{R}^d} f \geq 0$

Def: (Positive linear functional) : Let $\Sigma \subseteq L^1(\mathbb{R}^d, m)$ be a linear subspace. Then a linear functional $\Lambda : \Sigma \to \mathbb{C}$ is called positive if for $f \geq 0 \in \Sigma \Rightarrow \Lambda(f) \geq 0 \in \mathbb{C}$.

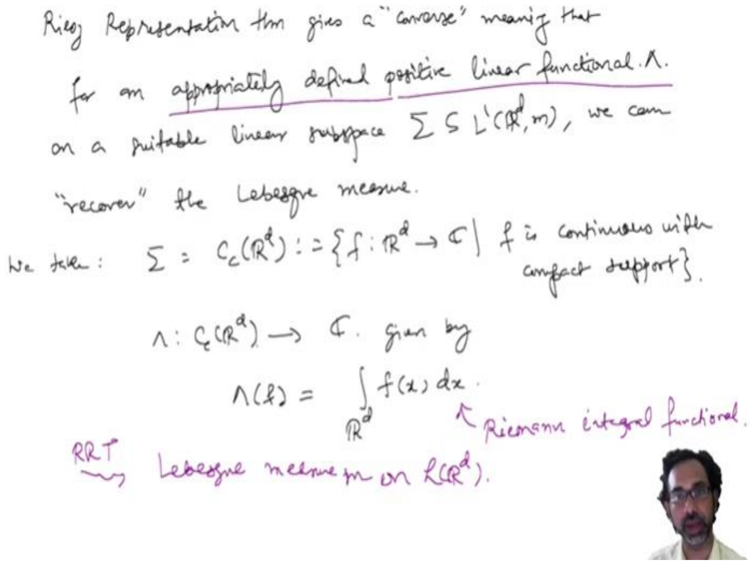
$\rightsquigarrow$ Lebesgue integral is a positive linear functional.

So let me out this as a remark that the Lebesgue integral functional is a linear functional on L1 Rd with the Lebesgue measurable. Now this Lebesgue integral also has the nice property of non-negativity meaning that if f is a positive function in L1 Rd then the integral of, f is a positive number rather a non-negative number. So this property of the Lebesgue integral functional makes it towards called a positive linear functional.

So let me define what is a functional linear functional let sigma be a sub-space of L1 Rd be a linear sub-space. Then a linear functional lambda which takes elements of sigma and gives you back complex numbers is called positive. If we have for f positive in sigma lambda f is positive in C. So this implies lambda f is positive in C. So with this definition we see that the Lebesgue integral is a positive linear functional.

So we have defined or rather we have obtained a positive linear function from our measure which was the Lebesgue measure on Rd.

(Refer Slide Time: 14:02)

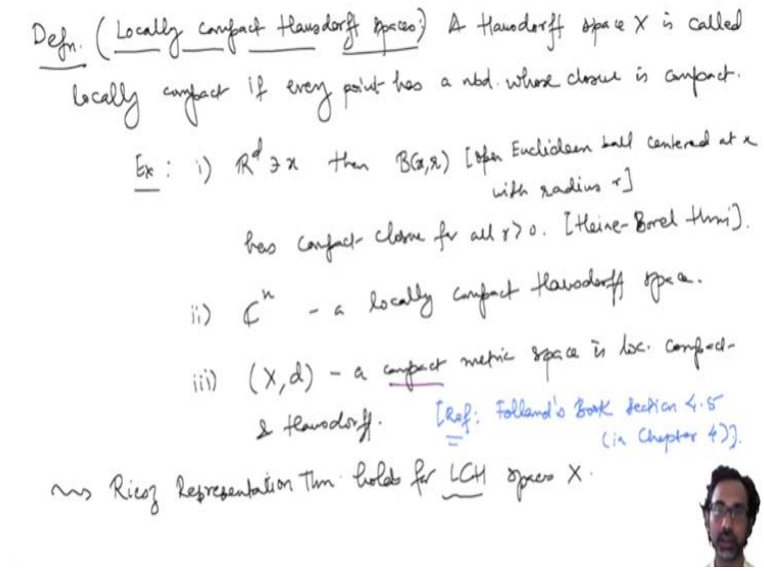


So the Riesz representation theorem gives a converse meaning that for an appropriately defined positive linear functional on a suitable. So let me call it lambda on suitable linear sub-space sigma of L1 Rd we can recover the Lebesgue measure. So in our case the appropriately defined positive linear functional lambda will in fact be the Riemann integration functional and sigma we take sigma to be the space of continuous compactly supported functions on Rd.

So these are functions Rd to C such that f is continuous with compact support and lambda when you define lambda from CC Rd to C given by lambda f equals integral over some big box B or let me write Rd here. Because f x d x so this is the Riemann integral functional which is also a positive linear functional. And when we define a Lambda with this we get back via the Riesz representation theorem we get back the Lebesgue measure.

So this is the yet another construction of Lebesgue measure and of course on we get back also the sigma algebra of Lebesgue measurable sets. So in fact the Riesz representation theorem works not only for Rd it also works for any locally compact Hausdorff space.

(Refer Slide Time: 17:26)



So let me recall what are locally compact Hausdorff spaces? So the definition for locally compact Hausdorff spaces is as follows. So a topological space X let us say we should already assume Hausdorff because this is something you know already. A Hausdorff space X is called locally compact if every point as a neighborhood whose closure is compact. So this is what a locally compact Hausdorff space is of course as an example we already have Rd this is due to the Heine-Borel theorem.

Because if X belongs to Rd then B x, r the open Euclidian ball centered at x with radius r this as compact closure for each r positive closure for all positive r. So this is due to the Heine-Borel theorem so we see that Rd is locally compact similarly the end dimensional complex vector space this is a locally compact Hausdorff space. And any compact matrix space is locally compact and Hausdorff of course this is Hausdorff because this is matrix space.

But since we are assuming compactness then it is also locally compact so locally compact Hausdorff spaces enjoy some nice properties and the main one is that there is that we will be interested in is that the Riesz representation theorem holds for locally compact Hausdorff. So I will abbreviate locally compact Hausdorff by LCH so it holds for locally compact Hausdorff spaces X.

Before I go to the statement and proof of phrase representation theorem we will have to recall some basic facts about locally compact Hausdorff spaces. Now good reference for this material that we shall use is a Folland's book section 4.5. So in chapter 4 section 4.5 this is on locally compact Hausdorff spaces and because we would have time to give all the proof so basic facts like Urysohn's lemma we have to go back and recall for yourself and see the details in this book.