Measure Theory
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Module No # 11
Lecture No # 55
Some criteria for reverse implications for modes of convergence

Now in the final topic regarding modes on convergence we will give some special criteria for reversing the chain of implications or some specific cases only.

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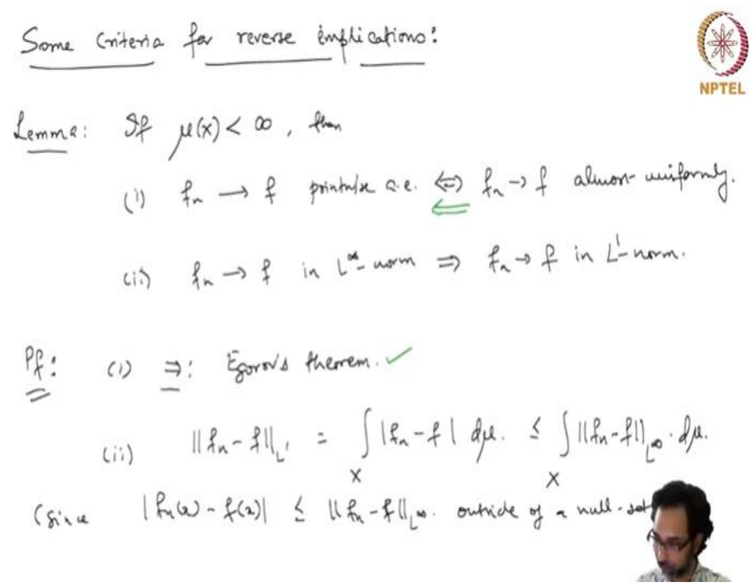

So the first lemma is that if the measure space is as finite measure the whole space has finite measure. Then the first part is that fn converges to f point wise almost everywhere is equivalent to fn converges into f almost uniformly. And secondly if fn converges to f in L infinity norm this implies that fn converges to f in the L1 norm as well. So of course this implication we had already seen and when so this lemma says that when the measure space has finite measure then the forward implications also hold.

And this is precisely for the first part this forward implication is precisely the statement of Agarose theorem that we have seen Agarose theorem which was precisely about almost uniform convergence when you have point wise almost everywhere convergence. So this we have already seen now for the second part if fn converges to a f in L infinity norm. Then if you take the L1 norm of the difference fn – f then this is nothing but the integral of fn –x fn – f d mu and this modulus fn – f is bounded above so this can be bounded above by the constant function fn – f L infinity d mu.

This is because mode fn x – fx is less than or equal to the L infinity norm outside of a null set. So since the integral does not see the null set we have this inequality.

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$$\int_X |f_n - f|\, d\mu \le \int_X \|f_n - f\|_{L^\infty}\, d\mu.$$

$$= \|f_n - f\|_{L^\infty} \cdot \underbrace{\mu(X)}_{<\infty} \to 0 \quad \text{as } n \to \infty.$$

And now we can take this so let me rewrite this d mu is less than or equal to norm of fn L infinity norm of fn –f d mu. And now this is a constant function over on X so we can take this out of the integral so this is nothing but fn – fL infinity norm times integral over X of the function 1 and this is nothing but the measure of X and this is finite. And so this goes to 0 as m goes to infinity so in the finite measure space setting we have seen a couple of reverse implications from point wise almost everywhere to almost uniform convergence and L infinity norm to L1 norm.

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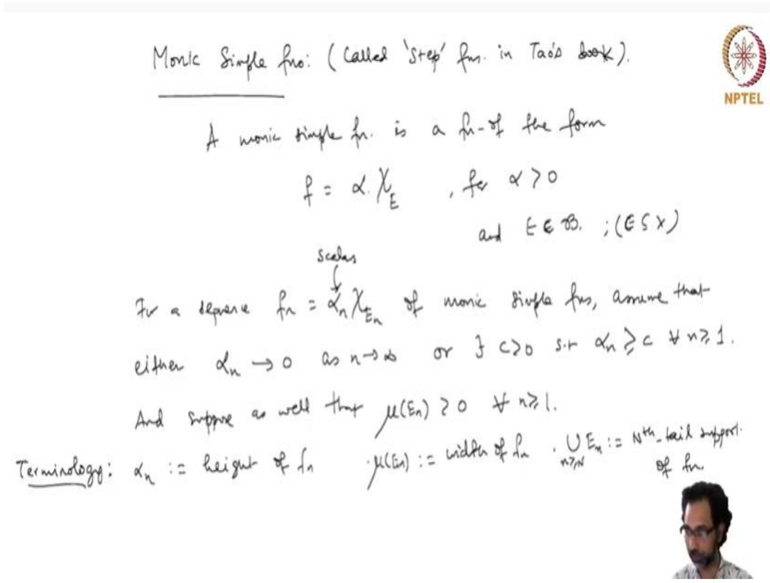

Now another special criteria is given by such so called Monic simple functions so these called step functions in Tao's books. So a Monic simple function is a function of the form f equals alpha times the indicate function of e for alpha strictly positive and E a measureable subset in X. So E is a measureable subset in X and now we will consider sequences of Monic simple functions. Now for a sequence fn equals alpha n chi En of Monic simple functions assume that either alpha n converges to 0 as n goes to infinity or their exist a positive constant c such that alpha n is bounded away from c.

So c is the lower bound for the alpha n's and we also suppose that as well that the measure of mu En's strictly positive for all n greater than or equal to 1. Now let us give some terminology so I will call alpha n height of fn mu En width of fn and if you take the union of En for n greater than equal to some N this is called the nth tail support of fn. So in analogy with functions define on the real line alpha n this coefficient alpha n's.

Of course I forgot to mention that these are all scalars and so these are heights of the function fn the measure of the sets en are called the widths of fn and the union of En for n greater than equal to N is called the nth tail support of fn. So these are just some terminology that is being used here so now we are ready to state our theorem.

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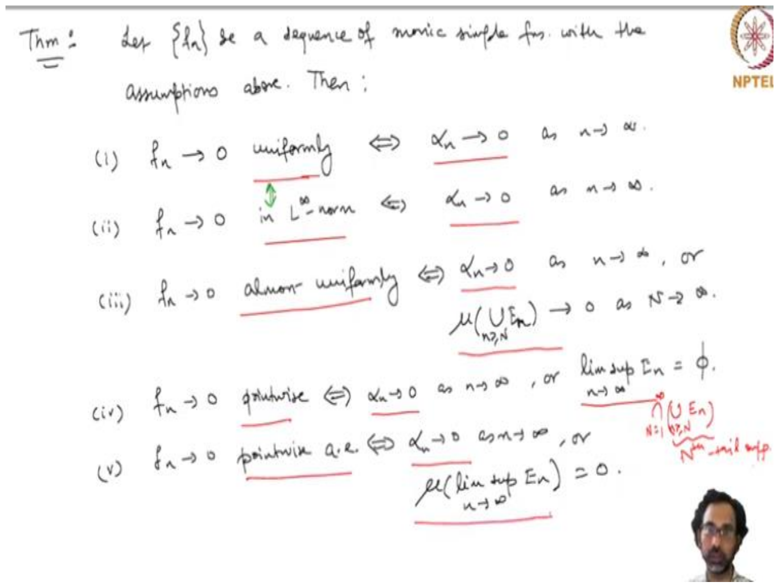

So this is the statement of the theorem and the theorem states give you characterization of this sequence of Monic simple functions to converge to 0 in all these 7 modes of convergence. In terms of these heights, widths and the behavior of the nth tail supports for these sequence of functions. So the first one says that for example if fn converges to 0 uniformly if and only if the heights alpha n as n goes to infinity.

Similarly if fn converges to 0 in L infinity norm if and only if the heights converge to 0 as n goes to infinity. So now we already see that we have seen that uniform convergence implies L infinity convergence but here we can revert this arrow and have an if and only if condition when uniform convergence and L infinity norm convergence are equivalent. So this is for Monic simple functions that satisfy these assumptions that we have stated.

Similarly fn converges to 0 almost uniformly if and only if either now we have 2 conditions either the height goes to 0 or the measure of the nth tail support goes to 0. So here we have an either or 2 conditions then we have point wise convergence if and only if either the height goes to 0 or the (()) (10:20) is the empty set. So note that this (()) (10:22) set is nothing but the intersection of all tail supports so this is precisely n = 1 to infinity union n greater than equal to N EN.

So this is the nth tail support so in terms of nth tail support we see that the intersection of all nth tail supports must be empty. And we can similarly state it for point wise almost everywhere convergence which says that either the height goes to 0 or the measure of the (()) (11:07) is 0.

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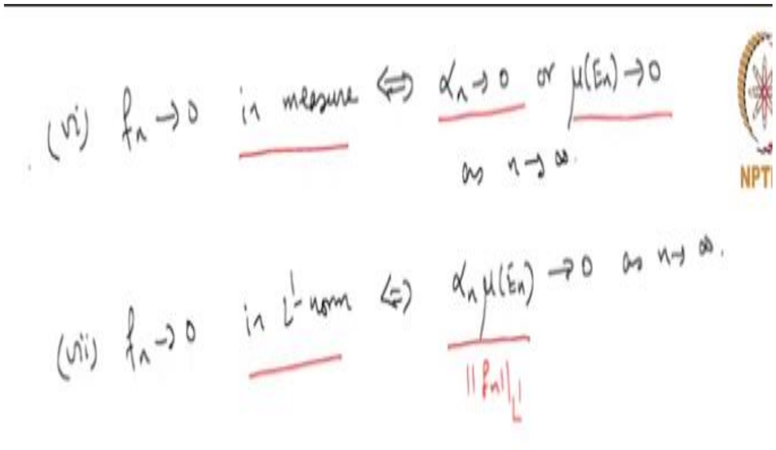

Similarly fn converges to 0 in measure if and only if the either the height goes to 0 or the width goes to 0 and finally fn converges to 0 in the L1 norm if and only if the product of the width and height go to 0 as n goes to infinity. Now this is actually quite easy to see the last one because this is simply the L1 norm of fn for each n so this one is almost trivial. I will only do the cases for convergence in measure so this one and convergence almost uniformly and the rest I will leave as an exercise.

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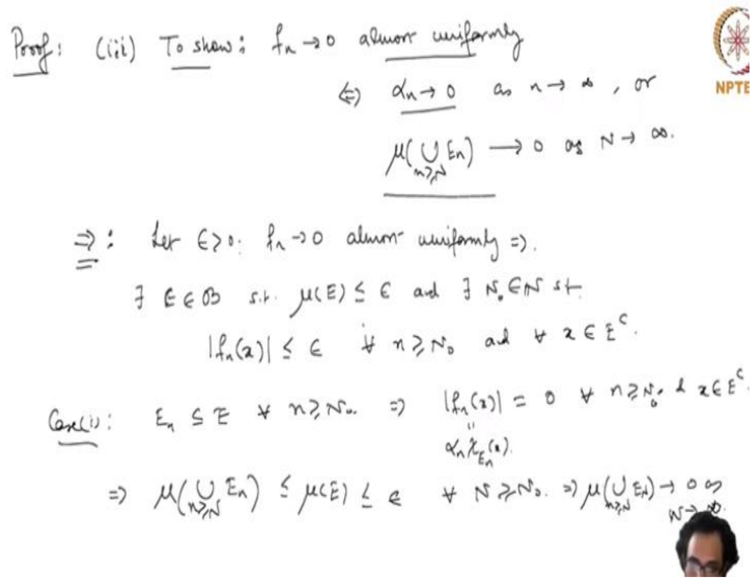

So let us see the proof for these 2 cases so for the third part we have to show that fn converges to 0 almost uniformly if and only if either the height goes to 0 or the measure of the nth tail support goes to 0. So let us start with for the forwarding implication let us start with epsilon greater than 0 and so because fn goes to 0 almost uniformly, which means that their exist a set E a measureable set E such that the measure of this set E is less than or equal to epsilon.

And one can choose an N naught in N such that we have modulus of fn x less than or equal to fn for all n greater than or equal to this threshold value N naught. And for all x in the complement of this exceptional set E which has measure less than or equal to epsilon. Now we can deal with 2 cases so that first case is that En is the sub set of E for all N greater than equal to N naught. So in this case this means that mod of fn x = 0 for all n greater than or equal to N naught and X in the complement of E because X does not belong to any of the sets En.

And because fn is equal to alpha n where indicate a function of En which is going to be 0 for all n greater than or equal to N naught. So this means that the measure of the union n greater than equal to N En is less than or equal to the measure of E which is less than or equal to epsilon for all N greater than or equal to N naught. And this means that the measure of the nth tail support goes to 0 as N goes to infinity. So in this first case where we have all these En's inside the set e then, we have automatically that the measure of the nth tail support goes to 0.

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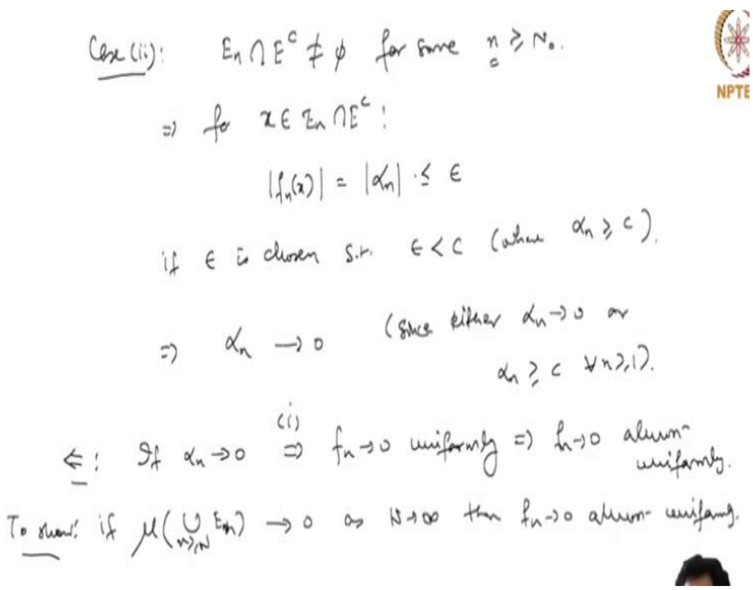

Now in the second case we have that x so En intersection E compliment is not empty for some n greater than or equal to N naught. So in this case this means that for x in En intersection e complement we have that the modulus of fn x is equal to the modulus of alpha n and then this is less than or equal to epsilon for all n greater than or equal to N naught because x belongs to En fn x is simply equal to alpha n.

And so sorry so this is not for all n this is for this particular value of n greater than or equal to N naught. But if epsilon is chosen less than this value c so such that epsilon is less than c where c was the lower bound for these alpha n's. Then this is a contradiction so this implies that alpha n must goes to 0 because since either alpha n goes to 0 or alpha n is greater than or equal to c for all n. Therefore either we have the measure of the nth tail support goes to 0 or alpha n goes to 0 the height goes to 0.

Now for the reverse implication if alpha n goes to 0 so this implies by the first part of our lemma that fn converges to 0 uniformly and this implies that fn converges to 0 almost uniformly. Now if the measure of the tail supports goes to 0 as N goes to infinity we have to show that so this show to show so this is the second case then fn goes to 0 almost uniform. So we will use this fact that this measure of the tail support goes to 0.

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Given $\epsilon > 0$, choose $N_0 \in \mathbb{N}$ s.t.

$$\mu\Big(\bigcup_{n \geq N} E_n\Big) \leq \epsilon \quad \forall N \geq N_0.$$

Set $E = \bigcup_{n \geq N_0} E_n$, then, $\forall N \geq N_0$, and $x \in E^c$

we have $|f_N(x)| = 0$.

$\Rightarrow f_n \to 0$ almost uniformly.

So given epsilon greater than 0 choose N naught such that the measure of the union n greater than or equal to N En is less than or equal to epsilon for all N greater than or equal to N naught. So now if you set E to be n greater than equal to N naught En then for all N greater than or equal to N naught and X in E complement we have where the modulus of fn x equals f N x equal to 0 because fn does to belong to any of this sets En sorry x does not belong to any of the sets En for n greater than equal to n naught.

So this fn x must be 0 and this implies that we have found that fn converges to 0 almost uniformly.

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(v) To show: $f_n \to 0$ in measure $\Leftrightarrow \alpha_n \to 0$ or $\mu(E_n) \to 0$ $n \to \infty$.

$\Leftarrow$: $\alpha_n \to 0 \Rightarrow f_n \to 0$ uniformly $\Rightarrow f_n \to 0$ in measure.

Suppose $\mu(E_n) \to 0$ as $n \to \infty$:

Given $\epsilon > 0$, $F_{n,\epsilon} := \{x \in X : |f_n(x)| \geq \epsilon > 0\}$

f_n can be either 0 or $a_n \Rightarrow x \in E_n \Rightarrow F_{n,\epsilon} \subseteq E_n$.

$\Rightarrow 0 \leq \mu(F_{n,\epsilon}) \leq \mu(E_n) \to 0$ as $n \to \infty$.

Squeeze thm. $\Rightarrow \mu(F_{n,\epsilon}) \to 0$ as $n \to \infty$.

$\Rightarrow f_n \to 0$ in measure.

So now we have left with the fifth part which is to show that fn converges to 0 in measure if and only if either alpha n goes to 0 or the measure of width goes to 0. So either the height goes to 0 or width goes to 0 as n goes to infinity. Let me start with the reverse implication now note that alpha n goes to o implies fn goes to 0 uniformly or even almost uniformly as we have seen and this implies fn goes to 0 in measure.

So suppose that rather that the width goes to 0 as n goes to infinity and in this case we have that if you set fn epsilon so given epsilon greater than 0. If you set fn epsilon to be the set of points such that the modulus of fn x greater than or equal to epsilon. So then because fn can either be 0 or an, this implies that since fn x is greater than or equal to epsilon and this is greater than 0 this implies that x belongs to En.

So this implies that fn epsilon is the subset of En and so the measure of fn epsilon is less than or equal to the measure of En so by this squeeze theorem this means that the measure of fn epsilon also goes to 0 as n goes to infinity because this goes to 0 and as n goes to infinity. And of course these are all positive so we get that the measure of fn epsilon goes to 0 as n goes to infinity and this is precisely the condition for fn going to 0 in measure.

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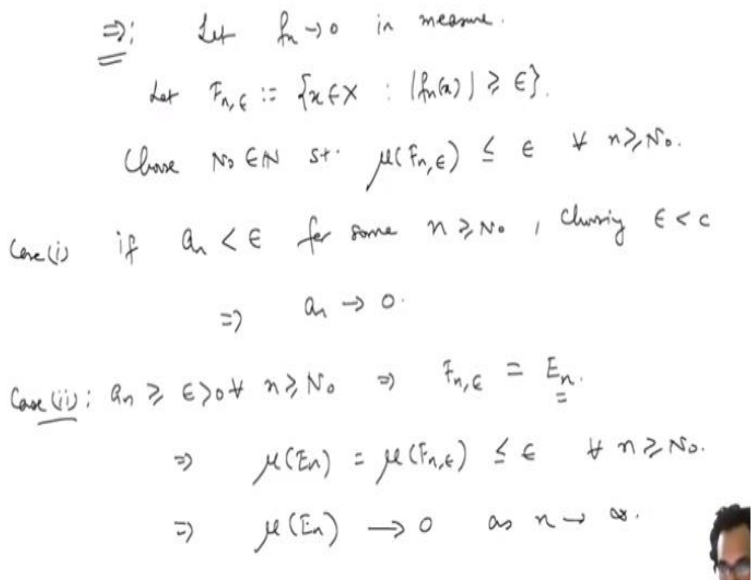

So this proves that reverse implication now for the forward implication let fn converge to 0 in measure so let again fn epsilon with the set of points such that fn x is greater than or equal to epsilon. Now choose N naught such that the measure of fn epsilon is less than or equal to epsilon for all n greater than or equal to N naught. So this is the condition for convergence in measure now if again we are going to separate in 2 cases.

So if an is less than epsilon for some n greater than equal to N naught choosing epsilon less than c implies that an converges to 0. So an, the other cases so this is case 1 these are the other case is that an is greater than or equal to epsilon for all n greater than or equal to N naught which means that the set fn epsilon is precisely En. Because this is positive and the set of points where fn is positive is precisely EN.

So this implies the measure of En is equal to the measure of fn epsilon and it is less than equal to epsilon for all n greater than or equal to N naught which means that the measures of these En's in the width goes to 0 and as n goes to infinity. So we see in the case of simple Monic functions there are some reverse implications that we can state.

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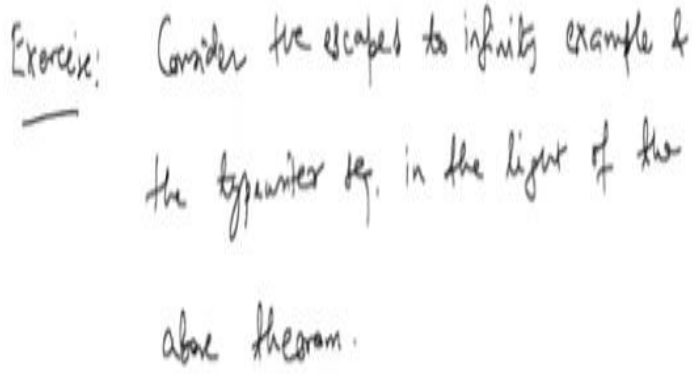

Now as an exercise consider the escape to infinity as example and the typewriter sequence in the light of the above theorem. So you will see immediately which one converges to 0 in which mode by using the above theorem and of course one as to also show the other parts of the theorem which I left to you as an exercise so this brings us to the end of the topic for modes of convergence.