Measure Theory
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Module No # 11
Lecture No # 54
Uniqueness of limits across various modes of convergence

Until now we have explored the various modes of convergence and we have seen how they are interrelated with each other and we have seen the various implications and the failure of the reversal of many of these implications. And in this lecture we will look at some limiting properties of the modes of convergence.

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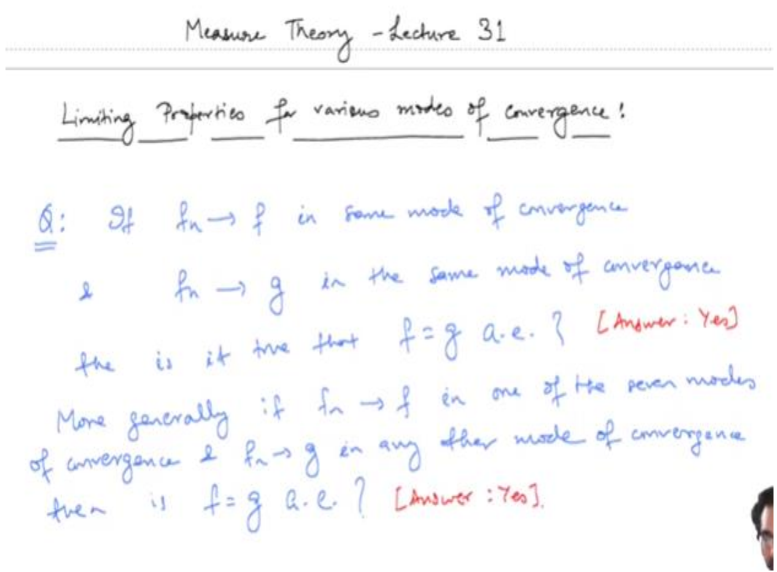

In particular we will ask the following question that if fn converges to f in some mode of convergence and the same sequence converges to other function g in the same mode of convergence as the above one in the same mode of convergence. When is it true that f = g and in fact we can ask more general question more generally if fn converges to f in 1 of the 7 modes of convergence and fn converges to g in any other mode of convergence.

Then is f = g so this is about the uniqueness property of the limits and we can first of all we can easily see that if we demand that f = g everywhere this is not going to be true even in the case of point wise almost everywhere convergence. So we have to relax our expectation little bit lower our expectation little bit and we only ask that f = g almost everywhere. Similarly here when if you have 2 different modes of convergence for fn going to f and fn going to g then f = g almost everywhere.

And they answer to both these questions is in the affirmative and we will prove this in this lecture.

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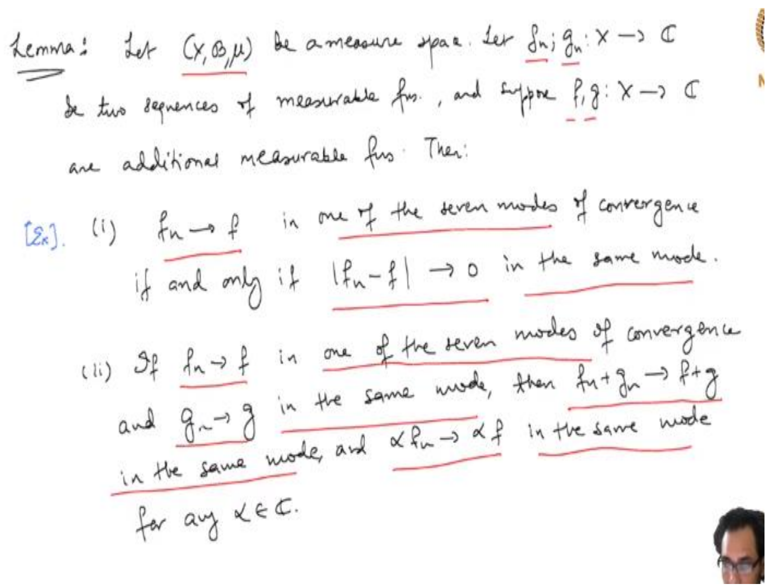

So let us begin with the following Lemma so here is the statement of the Lemma so we have a measure space x be mu and we have 2 sequences fn and gn of measureable functions on x complex measureable functions. And suppose that f and g are additional measureable functions then the first part says that fn converges to f in one of the 7 modes of convergence if and only if the modulus of fn – f converges to 0 in the same mode.

So this is the first part of the statement the second part is that if fn converges to f in one of the 7 modes again and gn converges to g in the same mode as fn then fn + gn converges to f + g in this same mode again. As well as alpha fn converges to alpha f in the same mode for any complex number alpha. So the first part is actually very easy but it is just that one as to check for each of this 7 modes that this is true but it is a quite easy and I leave it as an exercise.

So I will only prove the second part is essentially just the use of triangle inequality in its various forms and I will only prove it for convergence in measure.

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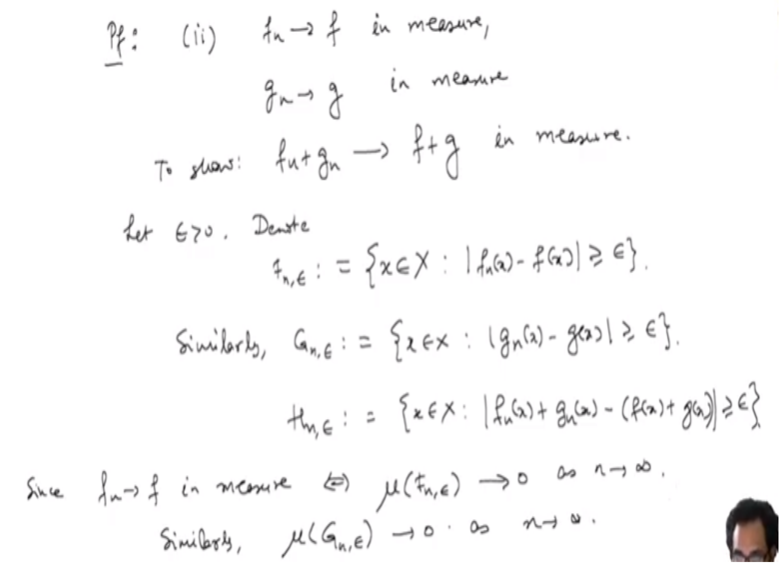

So let us look at the proof for 2 when fn converges to f in measure and gn converges to g n measure then I want to show that fn + gn converges to f + g in measure. To show this let epsilon be a positive number and denote by fn epsilon the following set this is the set of points in x such that modulus of fn x – fx is greater than or equal to epsilon. Similarly we define gn epsilon as the set of points in x such that gn x – gx is greater than or equal to epsilon and finally hn epsilon is the set of points in x such that fn x gn x – fx + gx is greater than or equal to epsilon.

So since fn converges to f in measure this is equivalent to saying that the measures of this sets fn epsilon goes to 0 as n goes to infinity. Similarly we have that measure of gn epsilon goes to 0 as n goes to infinity.

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We have to show: $\mu(H_{n,\epsilon}) \to 0$ as $n \to \infty$.

Note that if

$$\epsilon \le |(f_n(x) + g_n(x)) - (f(x) + g(x))|$$

$$\le \underbrace{|f_n(x) - f(x)|}_{\ge \epsilon/2} + \underbrace{|g_n(x) - g(x)|}_{\ge \epsilon/2}.$$

So, if $x \in H_{n,\epsilon} \Rightarrow x \in F_{n,\epsilon/2}$ or $x \in G_{n,\epsilon/2}$.

$$\Rightarrow H_{n,\epsilon} \subseteq F_{n,\epsilon/2} \cup G_{n,\epsilon/2}$$

$$\Rightarrow \mu(H_{n,\epsilon}) \le \mu(F_{n,\epsilon/2}) + \mu(G_{n,\epsilon/2}). \to 0 \text{ as } n \to \infty.$$

$$\Rightarrow f_n + g_n \to f + g \text{ in measure.}$$

And we have to show that the measure of so we have to show that the measure of hn epsilon goes to 0 as n goes to infinity. So now note that if epsilon is less than or equal to modulus of fn x + gn x – fx + gx then by usual triangle inequality we can write this as less than or equal to modulus of fn x – fx + gn x – gx. So if x belongs to hn epsilon then by this inequality this implies that x belongs to either fn epsilon by 2 or x belongs to gn epsilon by 2 meaning that at least one of these terms is greater than or equal to epsilon by 2.

So this means that hn epsilon is the subset of fn epsilon by 2 union gn epsilon by 2 and so the measure of hn epsilon is less than or equal to the sum of the measures fn epsilon by 2 + the measure of gn epsilon by 2 and the right hand side goes to 0 as n goes to infinity because both these terms go to 0. So this means that fn + gn converges to f + g in measure.

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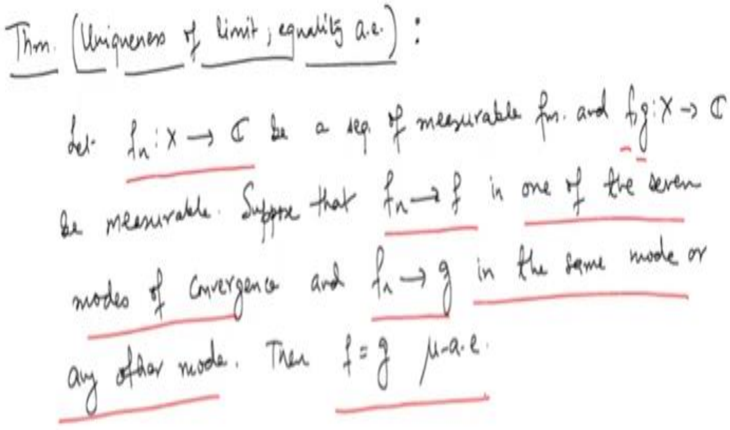

Equipped with this Lemma we can state the following theorem about uniqueness of limit with equality almost everywhere. And it says that if fn is the sequence of measureable functions and fn g are 2 additional measureable functions and now suppose that fn converges to f in 1 of the 7 modes of convergence and fn also converges to the function g in the same mode or in any other mode.

So for example fn can converge to f point wise almost everywhere and fn also converges to g it can be either point wise almost everywhere or in any other mode like converges in measure or L1. Then we have f = g mu almost everywhere so it means that as long as you have convergence to 2 different functions in any of the 7 modes. Then the limit of the functions will agree outside a null set so let us see a proof of this theorem.

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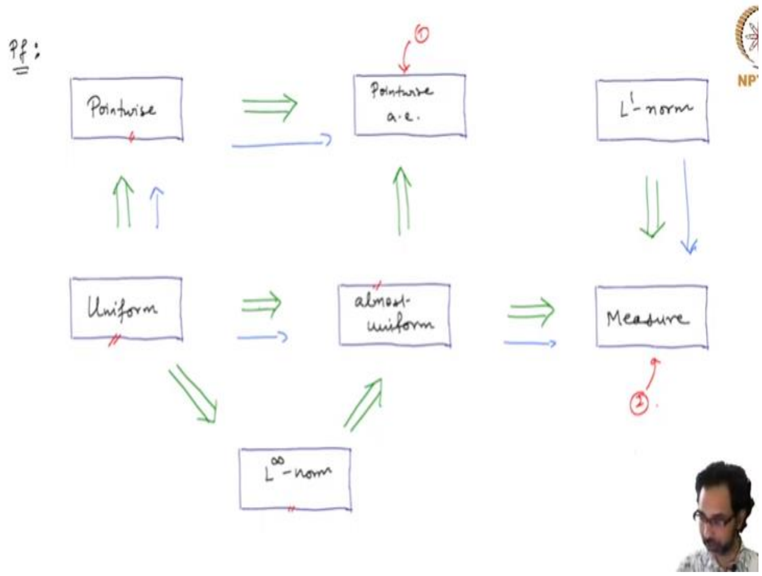

To start the proof it helpful to go back to our map of implication which will help us reduced the number cases to consider in our proof. So here notice that no matter where you start whether you start at uniform or L infinity or almost uniform or point wise you either end up if you follow the chain of implications then you either end up in point wise almost everywhere convergence either end up here or you end up in measure convergence.

So either 1 or 2 so for example if you start with uniform you can go to point wise and then to points wise almost everywhere similarly you can also go this way end up in measure convergence if you start with L1 you end up in measure convergence and so on. Because these arrows are not invertible in general so we cannot go backwards. So it is either point wise almost everywhere or convergence in measure.

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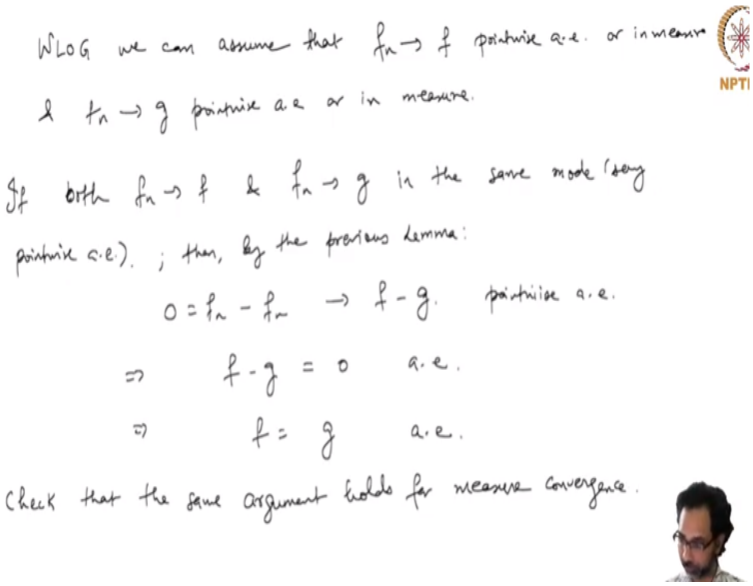

So this means that without loss of generality we can assume that fn converges to f point wise almost everywhere and gn converges to g or in measure. So fn converges to f point was almost everywhere or in measure and gn converges to g also point wise either point wise almost everywhere or in measure. So this is the first reduction that we can make because our chain of implications that leads us to these 2 modes of convergence.

Now if both sorry this should be fn if both fn converge to f and fn converges to g in the same mode say point wise almost everywhere then by our Lemma by the previous Lemma we have that fn – fn. So we choose the 2 sequences fn and –fn and we add them so you get the 0 sequence this converges to f – g so this is by our previous Lemma because if you add 2 sequences that converges to fn g and then the some converges to f + g.

And so we have this but this means that f – g so this is point wise almost everywhere but since this is a constant sequence this means that f – g equals to 0 almost everywhere. So this means that f = g almost everywhere and we have done similarly if you are considering measure convergence one can use the similar argument to conclude that f = g almost everywhere. So check that the same argument holds for measure convergence. So we are only left to prove we can use the symmetry of the equality almost everywhere.

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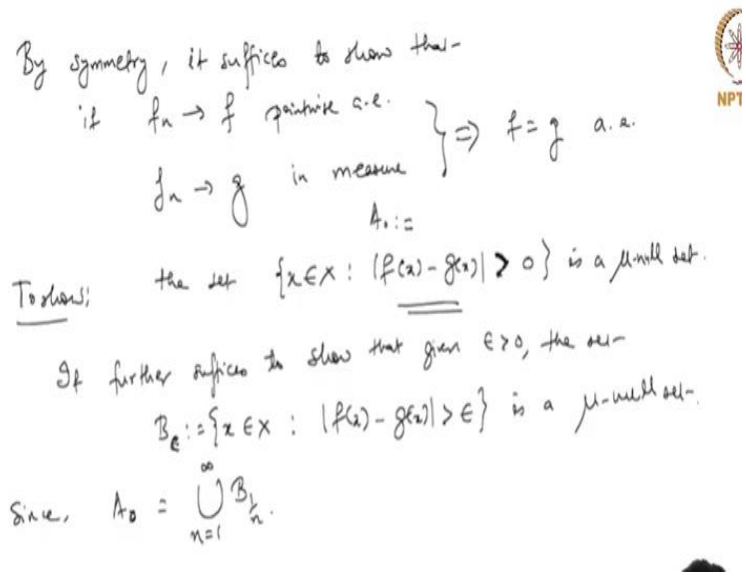

So by symmetry it suffices to show that if fn converges to f point wise almost everywhere and fn converges to g in measure then these 2 things implies that f = g almost everywhere. So by reversing the rows of g and f you can get the reverse case where fn converges to fn measure and fn converges to g point wise then also this should work. So to show this it further suffices so we have to show first that the set x in X such that mode fx – gx is greater than 0 strictly greater than 0 is a mu null set.

So if you want to show this it further suffices to show that given any epsilon greater than 0 the set x in X such that modulus of fx – gx is greater than epsilon is a mu null set. Because if we write this as a0 and if we write this as b epsilon then a0 is the union countable union n = 1 to infinity B1 / n. So we can describe this set n naught where fx – gx are has it strictly positive the modulus is strictly positive.

With respects to these sets b epsilon by choosing epsilon to be 1 over n and taking the union and if each of these B1 / n are mu null then the union is mu null. So we have reduce it to prove in that this set B epsilon is a mu null set.

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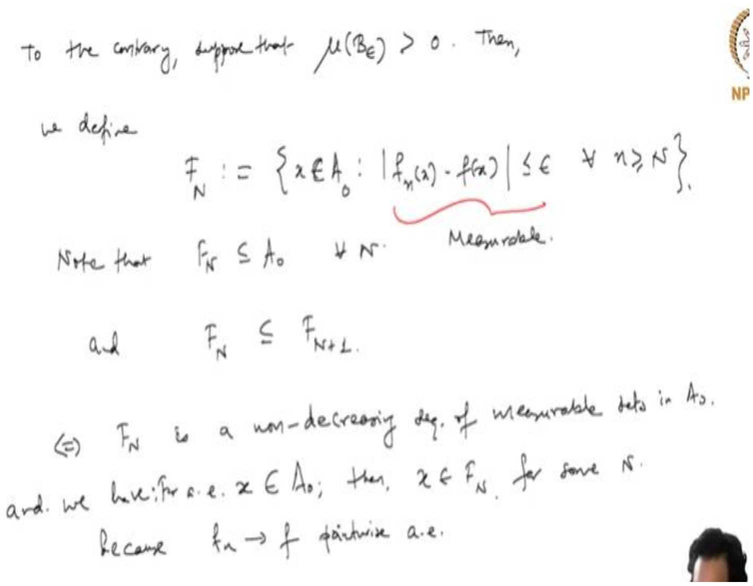

So let us see to the contrary we suppose the measure of B epsilon is strictly positive. Then we define the following set we define fn to be the set of points in X such that f N fn – x fx the modulus is less than or equal to epsilon for all n greater than equal to n. So sorry here I should take x in A0 so these are all subsets of A0 so note that fN is the subset of A0 by construction for all n and fN is the subset of fN + 1.

Because if this condition holds true for all n greater than equal to N then it also holds to for all N greater than or equal to N + 1. So fN is the subset of fN + 1 and therefore this implies that fN is a non-decreasing so this is equivalent to saying that fN is non-decreasing sequence of measureable sets. So these are all measureable sets each fN is measureable this is measureable because fN and f are both measurable functions.

So this set is measureable so fN is non-decreasing sequence of measureable sets in A naught so these are measureable subsets of A naught. And we also have that if x belongs to x any point or rather let us take A naught then X belongs to fn for sum n. Because fn converges to f point wise almost everywhere so outside the null set. So for almost every x for almost very X in A naught X belongs to some fN for some N due to the point wise almost everywhere convergence of, f as the functions fn 2 the function f.

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$\Rightarrow \quad A_0 = \bigcup_{N=1}^{\infty} F_N.$

therefore by UMCT ; $\mu(F_N) \to \mu(A_0)$. as $N \to \infty$.

Since $\underline{\mu(A_0)} > 0 \Rightarrow \mu(F_N) > 0$ for some $N \in \mathbb{N}$.

Same argument shows that $\mu(\underbrace{F_N \cap B_\epsilon}_{F_{N,\epsilon}}) > 0$ for some $N \in \mathbb{N}$.

for $x \in F_{N,\epsilon} = F_N \cap \underline{B_\epsilon}$.

$$\epsilon < |f(x) - g(x)| \le \underbrace{|f_n(x) - f(x)|}_{\le \epsilon/2} + \underbrace{|f_n(x) - g(x)|}_{> \epsilon/2}$$

$\Rightarrow$ On a set of positive measure, $|f_n(x) - g(x)| \geq \epsilon/2 \quad \forall n \geq N$.

So this means that A naught can be written as the union of all this sets fn because it is a non-decreasing sequence of subsets and A naught is the subset of this union. So A naught is equal to the union and therefore by upward monotone convergence theorem the measure of fn converges to the measure of A naught as n goes to infinity so since the measure of A naught is supposed to be strictly positive.

This implies that the measure of fn is strictly positive for some n in the natural numbers because this is a sequence of this sequence converges to the value mu A naught and so mu fn is greater k than 0. Now the same argument shows that the measure of fn intersection B epsilon is greater than 0 for some n in the natural numbers. Because you can simply replace A0 here by B epsilon and B epsilon we have assumed to have positive measure.

So all the arguments that we have mentioned go through this new set fN epsilon as positive measure. Now if we write for x in fN epsilon so this is fN intersection B epsilon so this means that modulus of fx – gx is greater than epsilon. So this is the part this part follows from the fact that x belongs to B epsilon. And now if I use the triangle inequality here so you can write this bound it above by fn x – fx + fn x – gx and because x belongs to fN then this part is less than or equal to epsilon.

So in fact I would like it to have value less than epsilon by 2 so let me go back and change it a little bit so I would like it here to be less than or epsilon by 2. And so if this is less than or equal to epsilon by 2 and the sum is greater than epsilon so this means that this is the second term is strictly greater than epsilon by 2. This means that on a set of positive measure we have fn x – gx is greater than epsilon by 2 for all n greater than equal to n.

So this holds for all n greater than equal to N this is how we define our set f N which means that fn cannot converge to g in measure.

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$\Rightarrow$ $f_n \not\to g$ in measure (a contradiction).

So $f = g$ μ a.e.

So this means that fn cannot converge to g in measure which is a contradiction. So f = g mu almost everywhere so this finishes the proof of our uniqueness result comparing limits when you have when the same sequence converges to 2 different limits in different modes or even the same mode of convergence then the 2 functions are equal outside of the null set.