Measure Theory
Prof. Indrava Roy
Department of Mathematics
Indian Institute of Mathematical Science

Module No # 11
Lecture No # 53
Implication map for modes of convergence with various examples

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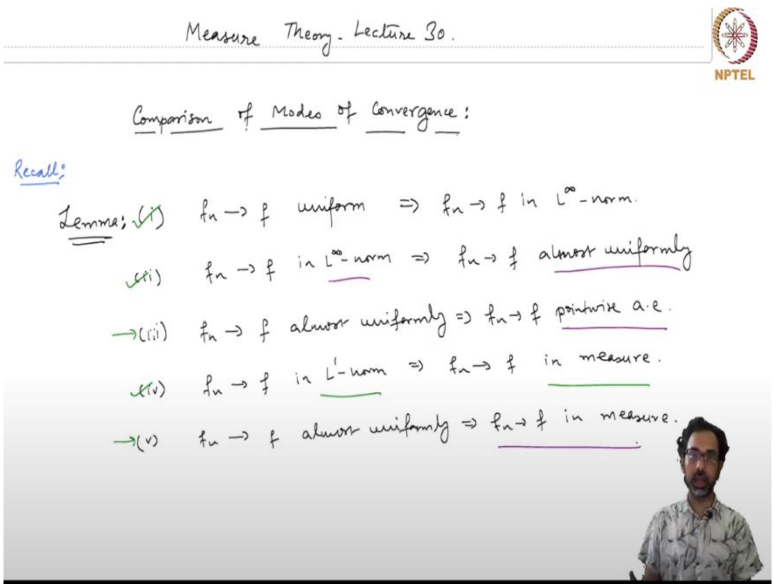

So in this lecture we continue our comparison for the various modes of convergence that we have defined. And we want to explore their interdependence or interrelationship between these modes of convergence. So we recall that we stated this Lemma and we proved some parts of this Lemma. So, the first one was that uniform convergence implies L infinity convergence so we have already seen this. This is actually quite trivial.

We have also seen that L infinity convergence implies all convergence this is also this something we have proved in our last lecture. And we have seen that L1 convergence implies convergence in measure. So we have already proved this part. So in this lecture first of all we will prove this part and this part third and fifth where almost uniform convergence implies both point wise convergence almost everywhere as well as convergence in measure.

So the third part is convergence almost point wise almost everywhere. And the fifth part is convergence in measure. So let us look at the proof for these parts.

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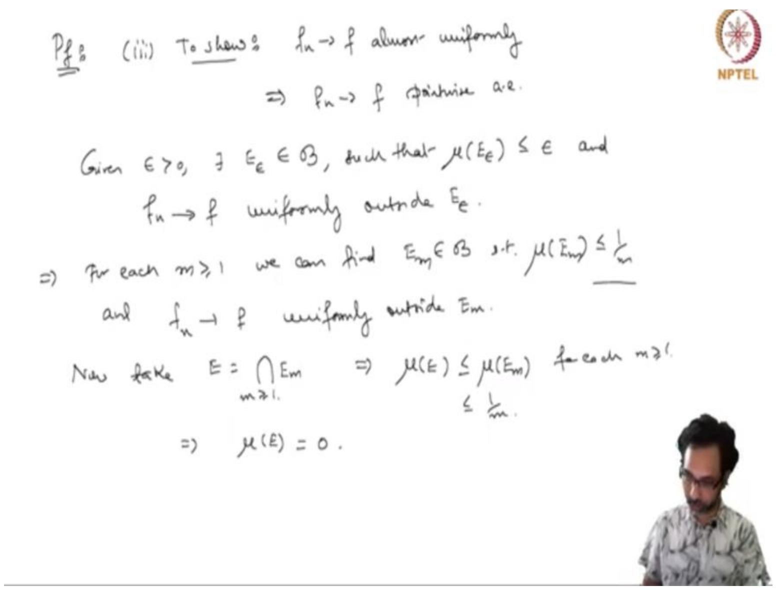

So we want to show that fn convergence to have almost uniformly implies fn converges to f point wise almost everywhere. So let us start with the definition of almost uniform convergence which is that given epsilon greater than 0. There exist a measurable set E epsilon such that measure of E epsilon is less than or equal to epsilon and fn converges to f uniformly outside of E epsilon. So this is the definition of almost uniform convergence.

So given this definition we can find for each n greater than equal to 1 we can find a set En a measurable set En such that the measure of En is less than or equal to 1 over n. And fn converges to f so let me write here m rather than n because I am using n for the sequence fn Em and E 1 over m here. So fn converges to f uniformly outside Em. So for each m we have found a measurable set Em which as measure less than or equal to 1 over m.

And fn convergence to f uniformly outside of En. So now we can take E to be the intersection of all these Em's. So this means that the measure of E is less than or equal to the measure of Em for each m and this is less than or equal to 1 over m. And since m can be taken as large as possible this implies that the measure of E is equal to 0.

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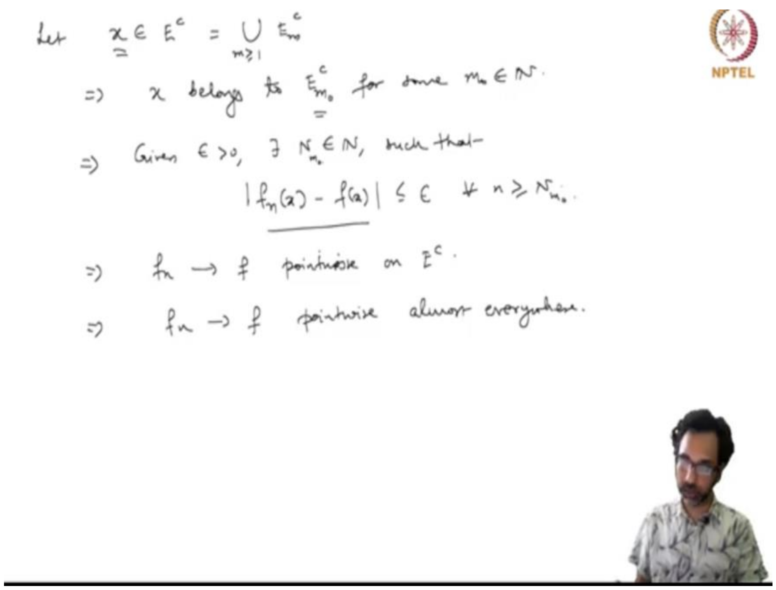

Now let see what happens on a point x which belongs to the complement of E. So the compliment of E is nothing but the union m greater than or equal to 1 of the complements of Em by definition. This means that x belongs to Em naught compliment for some m naught in N. So it belongs to one and the compliment of one of this Em's. And so this implies by the construction of our Em's that the modulus so given epsilon greater than 0 there exists a capital N belonging to N such that modulus of fn x - fx is less than or equal to epsilon for all n greater than or equal to n.

For this particular x that we have chosen here because it belongs to Em naught compliment. So N here will depend on your m naught. So this means that fn converges to f point wise on E compliment and because of the measure of E is 0 which means that fn converges to f point wise almost everywhere. So this shows that almost uniform convergence implies convergence point wise everywhere.

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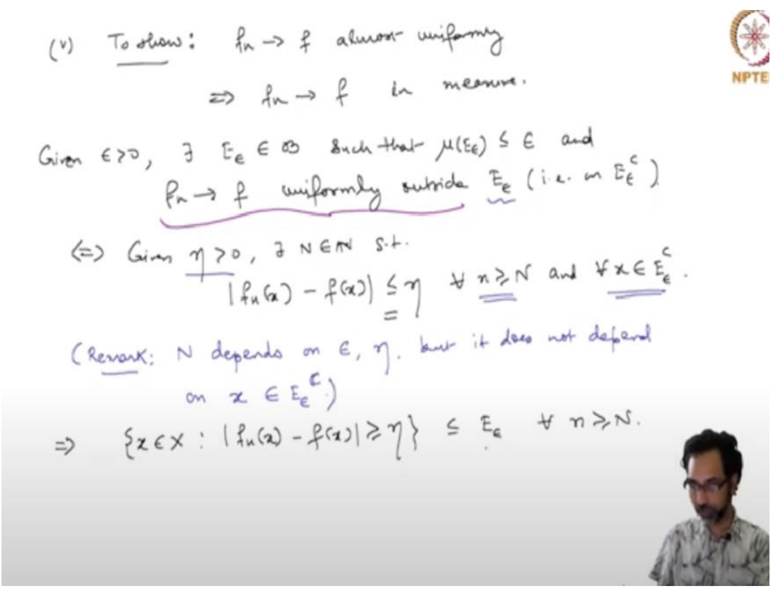

Now we come to the fifth part which is that we have to show that fn converges to f almost uniformly implies that fn converges to f in measure. So given epsilon greater than 0 let me again repeat the definition of almost uniform convergence their exist a set E epsilon B in the sigma algebra B such that the measure of E epsilon is less than or equal to epsilon and fn converges to f uniformly outside E epsilon outside that set E epsilon.

So this means that on E epsilon compliment. So this means if we write this if we unpack this part fn converges to f uniformly outside E epsilon. So what does it mean? This means that given eta now greater than 0 this I another arbitrary positive number. There exist a N such that the modulus of fn x minus f x is less than or equal to eta for all N greater than equal to n and for all x in E epsilon compliment.

So I just, written down what it what it means for fn to converge uniformly to f on E epsilon compliment. So this is the statement that given any eta now greater than 0 there exist a capital m such that the difference fn x - fx is less than or equal to eta for all n greater than equal to n and for all x in E epsilon compliment. So this n here remark that this n depends on first of all it depends on epsilon because this uniform convergence is outside E epsilon which in turn depends on epsilon.

So it depends on epsilon it also depends on eta. But it does not depend on x in E epsilon compliment. So this means that this N this threshold N works for any x in E epsilon compliment ok. So now we have this so this implies that the set of points in x such that the modulus of fn x - f x greater than or equal to eta. Is therefore a subset of E epsilon for all n greater than or equal to this N because outside E epsilon it must be less than or equal to eta right.

For all n greater than or equal to N. So this is the kind of set we want to estimate when we are talking of convergence in measure. So therefore the measure of this set is bounded above by the measure of E epsilon.

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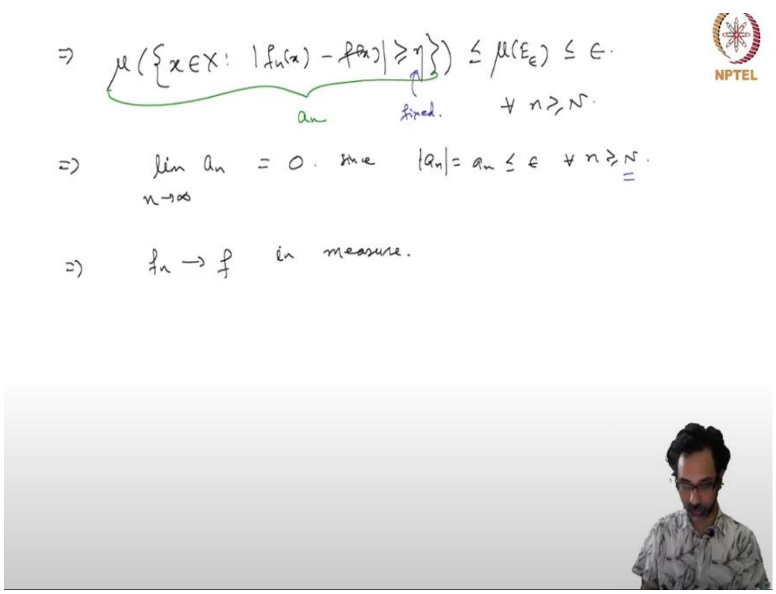

So this implies that the measure of the set of points such that fn x - fx is greater than equal to eta is less than or equal to E epsilon which is less than or equal to epsilon. This is for all n greater than equal to N. This means that the limit of these sequence of terms if they call it an then the limit of an as n goes to infinity is 0. Because since modulus of n which is the same as an which is positive this is less than or equal to epsilon or for n greater than or equal to N.

So here in the sequence an this eta is fixed and so we have proved that fn convergence to f in measure. So this completes our proof for our easy implication 1 to 5 for all these implications. So it is actually quite useful to have a picture in mind.

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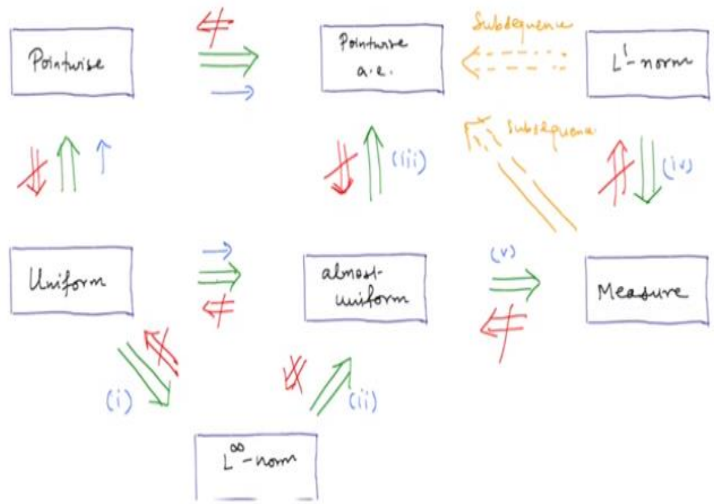

So this is a map of implications of the various modes of convergence we have seen. So there are seven in total. We have point wise almost everywhere then we have almost uniform measure L infinity norm convergence in L1 norm. So uniform implies point wise is quite basic and so is point wise implies point wise almost everywhere. So this part and this part is trivial. Now uniform implies L infinity norm this is this was part 1 of our Lemma.

Then L infinity norm implies almost uniform convergence this was part 2 of our Lemma. Then part 3 was almost uniform implies point wise almost everywhere convergence. Part 4 was L1 implies measure convergence. And part 5 was almost uniform implies measure. So of course if uniform implies Ln infinity and L infinity implies almost uniform. Then this is also taken care of. So we have this set of implications. Now we will go back to our escapes to infinity and see that we are not able to invert most of these implications.

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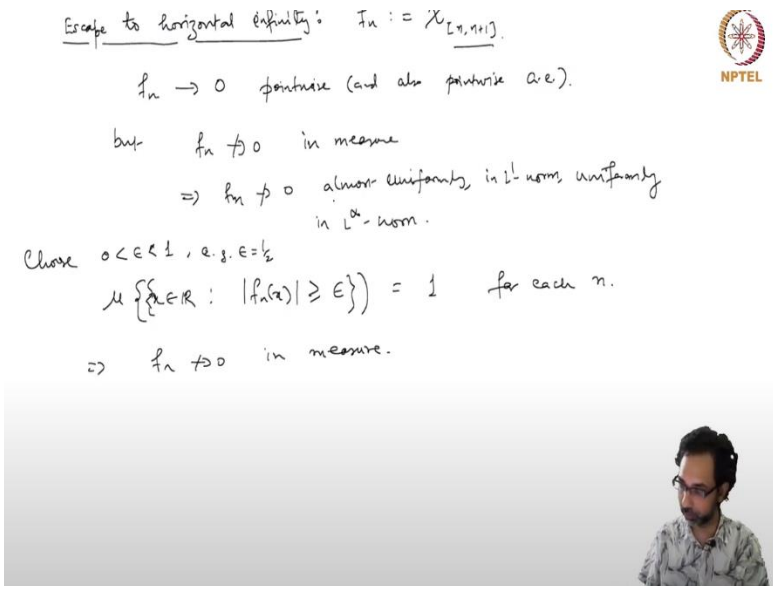

So let see the escape to horizontal infinity example what we had so this was the sequence of the indicate functions for the interval n, n+1. And we showed that fn converges to 0 point wise and also point wise almost everywhere. But let us show that fn does not converge to 0 in measure. And so this would imply that fn does not converges to 0. Almost uniformly in L1 norm uniformly in L infinity norm as well because the measure convergence is implied by almost uniform uniformly L infinity as well as L1.

So if it does not converge in measure then it does not converge in any of these 4 modes. So let us see why it does not converge 0 in measure this is simply because the measure of those points x in R such that the modulus of fn x is greater than or equal to epsilon. For example I can take choose epsilon between strictly between 0 and 1 for example epsilon can be taken to be half. And then the modulus of f and x greater than or equal to epsilon the measure of such sets is precisely equal to 1 for each n.

Therefore it cannot go to 0 and so fn does not converge to 0 in measure. So this is an example which shows that convergence point wise almost everywhere does not imply almost uniform convergence. And so we do not have this reverse implication.

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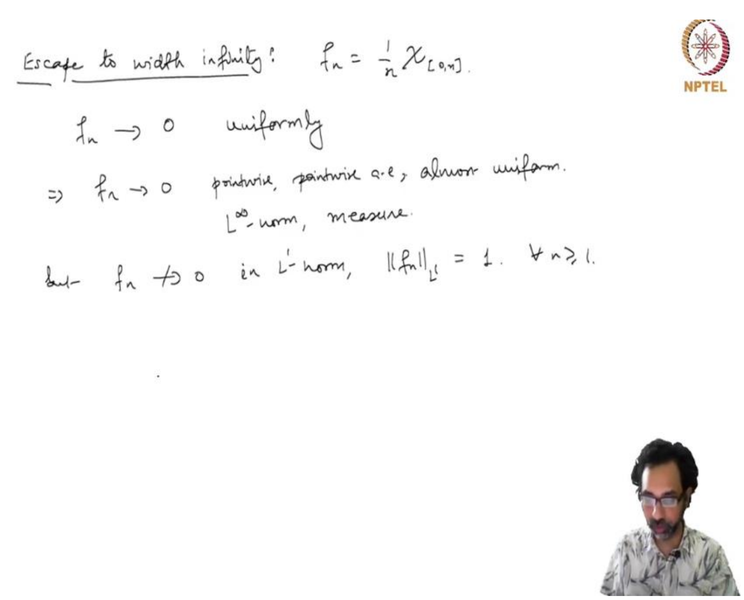

Now let us look at escape to width infinity again which was the sequence of functions defined by 1 over n indicative function of the interval 0 to, n. So we had seen that fn converges to 0. Uniformly and this implies that fn converges to 0 point wise almost to everywhere. And then it also implies almost uniform convergence and it also implies L infinity convergence and it also implies convergence in measure.

But fn does not converge to 0 in L1 norm because the L1 norm is fixed at the value 1 this is for all n. So we see that measure convergence in general does not imply L1 convergence. So in fact this implication also does not hold for our escape to width infinity example.

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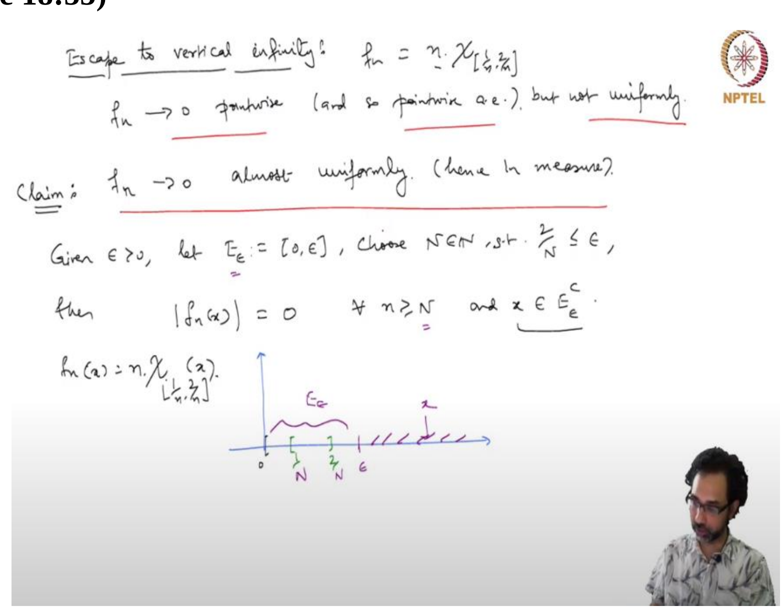

Another interesting example was given by escape 2 vertical infinity where we had the sequence of function fn given by n times the indicator function of interval 1 over n to 2 over n. So we have seen that fn converges to f0 point wise and so point wise was almost everywhere but it does not converge uniformly to 0. Now we claim that in fact fn convergence to 0 almost uniformly and hence also in measure. So let us see why this is true.

So let given epsilon greater than 0 let E epsilon be the set with the interval 0 to epsilon. So now if we choose N such that 1 over n is less than or equal to epsilon then the modulus of fn x is actually going to b 0 for all n greater than or equal to n and x belonging to the compliment of this interval 0 epsilon. So what is happening here so, we had these intervals 1 over n to 2 over n and we have some threshold epsilon let say this is epsilon.

So this n is large enough so for example if we take 1 over N 2 over N. So actually here take 2 over N. So, that the entire interval is between 0 and epsilon. So then this interval 1 over n to 2 over n becomes inside E epsilon and so because fn x is n times the indicative function of 1 over n 2 over n. Then for any x outside this interval so any x here in this region so for example x could be here then the indicator function is gives the gives you the value 0 because your interval 1 over N 2 over N is inside epsilon.

So this part is E epsilon. So outside E epsilon you are going to get 0 and this happens for all n large enough. So we see that this is almost uniform convergence where our exceptional set it can be chosen to be the interval 0 to epsilon. And so it also converges to 0 in measure because almost uniform convergence implies measure convergence.

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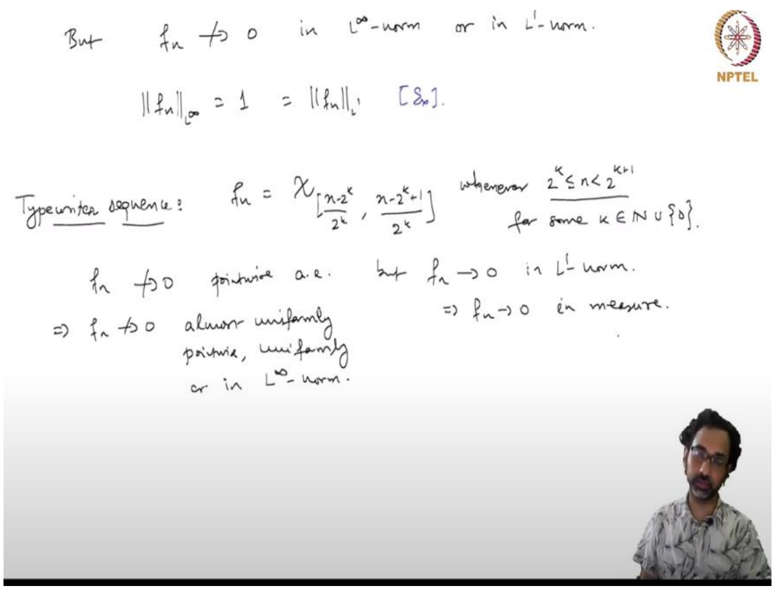

But fn does not converge to 0 in L infinity norm or in L1 norm. The L infinity norm for each fn is actually 1 and so is the L1 norm for each fn. So I leave it to you as an exercise to check that both these equalities are true. So we have seen that in this case fn convergence point wise to 0 point wise almost everywhere to 0 almost uniformly and in measure but not in any other way.

So if we go back to our map. So here this part this implication does not hold. Of course also this implication does not hold and also this implication does not hold. The final example was is the typewriter sequence which we saw that the sequence was taken to be indicator function of the interval n - 2 to the power k over 2 to the power k2 n - 2 to the power k + 1 over 2 to the power k whenever n lies between 2 to the power k and 2 to the power k + 1 for some k which could be either a natural number or 0.

So this is the typewriter sequence that matches back and forth on the interval 0 and 1. And fn we have seen that does not converge to 0 point wise almost everywhere but fn converges to 0 in L1 norm. So because fn converges to 0 in L1 now this implies that fn converges to 0 in measure as well and because fn does not converge to 0 point wise almost everywhere. This implies that fn does not converge to 0 almost uniformly point wise uniformly or in L infinity norm. So it violates all other modes of convergence except L1 and measure.

So if we go back to the implication map so we see that even this one does not hold in general. And of course we have seen that point wise almost everywhere or in L infinity norm does not imply uniform in general. This is because just as we did for point wise almost everywhere does not imply point wise we can change the sequence on a set of measure 0. So that it violates uniform convergence or point was convergence.

So we can we have given examples where none of these or all of these implication failed to be inverted. Now what I can say for a final few words for this map is that we have seen though that convergence in L1 norm implies that there is a subsequence which converges point wise almost everywhere. So this is extracting a subsequent. And in fact one can strengthen this a little bit and even for measure there is a subsequence which converge point wise almost everywhere.

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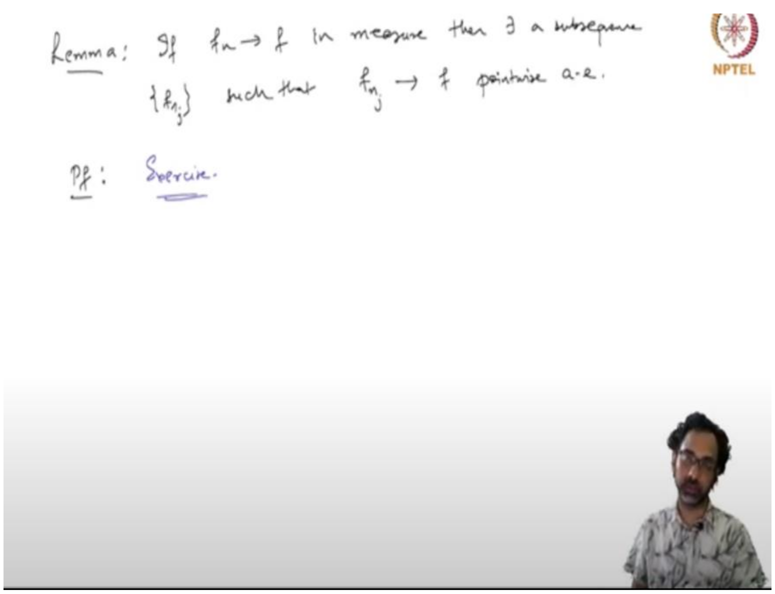

So I will just take this as a Lemma and leave it to you as an exercise that if fn converges to f in measure then there exist a subsequence fn j such that fn j converges to f point wise almost everywhere. So I leave this proof as an exercise. So just try to mimic the proof for the case when we had L1 convergence and we extracted a subsequence by taking a fast enough subsequence and then that subsequence converges point wise almost everywhere and a similar argument works for this case also.