Measure Theory
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Module No # 11
Lecture No # 52
Easy implications from one mode of convergence to another

Now we would like to compare all this modes of convergence these 7 modes we have defined until now.

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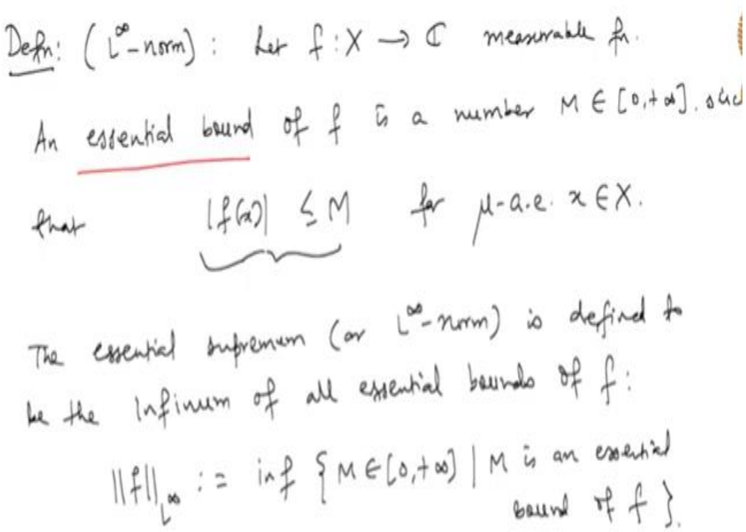

But before that let me give another definition and this is of the L infinity norm so suppose that f is a complex value measureable function on x. So the first definition is an essential supremum or essential bound of, f is a number positive number M such that mod fx is less than or equal to M for mu almost everywhere X. So this bound holds for almost every X so this is outside of a null set in X.

So any such positive constant M for which this condition is satisfied is called an essential bound of, f. Now the essential supremum or L infinity norm is defined to be the infimum or essential bounds of f which means that we can denote it as f L infinity like this. So if you want you can also write it as measure mu on it but we will usually drop this measure mu from the notation and simply write it as L infinity like this.

And this is by definition the infimum of all numbers so I am assuming also M can be plus infinity so here also we can take M to be plus infinity because your function may not be bounded on may not have any essential bound in that case M will be plus infinity. So here it is the infimum of M lying in this extended non negative real's such that M is an essential bound of, f as defined above so this is the L infinity norm.

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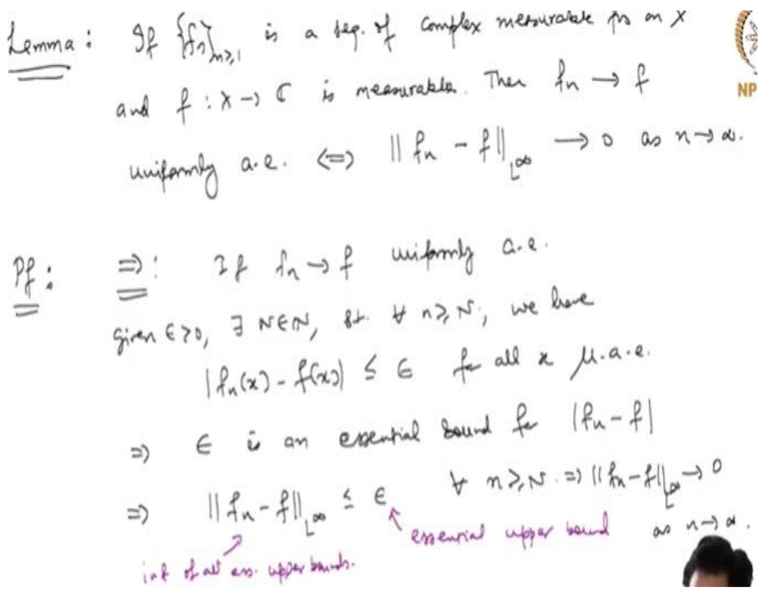

So one immediate result is the following so if fn is a sequence of complex measureable functions on X and f is another complex measureable function is measureable then fn converges to f uniformly almost everywhere. So what we call also essentially uniformly converges essentially uniformly if and only if this norm L infinity norm are the difference L infinity goes to 0 as n goes to infinity.

So we see that convergence in the L infinity norm is equivalent to convergence uniformly almost everywhere or essentially uniform. So let us a quick proof so we can recall the definition of convergence uniformly almost everywhere. So fn converges to f uniformly almost everywhere if given epsilon greater than 0 their exist a N such that so I missed something which is this is for all n greater than equal to N this conditions holds for mu almost every x.

So to show the forward implication we have that if fn converges to f uniformly almost everywhere given epsilon greater than 0 their exist a N such that for all n greater than or equal to N we have mode of fn x – fx is less than or equal to epsilon for all x mu almost everywhere. So outside of a mu null set this relation holds for all n greater than or equal to N. Now this means that by definition of the L of an essential bound this means that epsilon is an essential bound for the function fn – f mod.

So absolute value of fn – f so this is a this implies that the L infinity norm of fn – f so we can put modulus if you like but it would be the same thing. So it says that it implies that the L infinity norm of fn-f is less than or equal to epsilon because on the right hand side this is an essential upper bound and on the left hand side this is infimum of all essential upper bounds. So this L infinity this difference as L infinity norm bounded above by epsilon and this is valid for all n greater than equal to N.

And this implies that this L infinity norm of the difference goes to 0 as n goes to infinity so this is the forward implication and I leave the reverse implication as an exercise.

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$\Leftarrow$: Left as an exercise.

So prove the reverse implication left as an exercise which is essentially the same thing as we have seen in the forward implication as an exercise. So we see that uniform convergence almost everywhere is the same as convergence in the L infinity norm.

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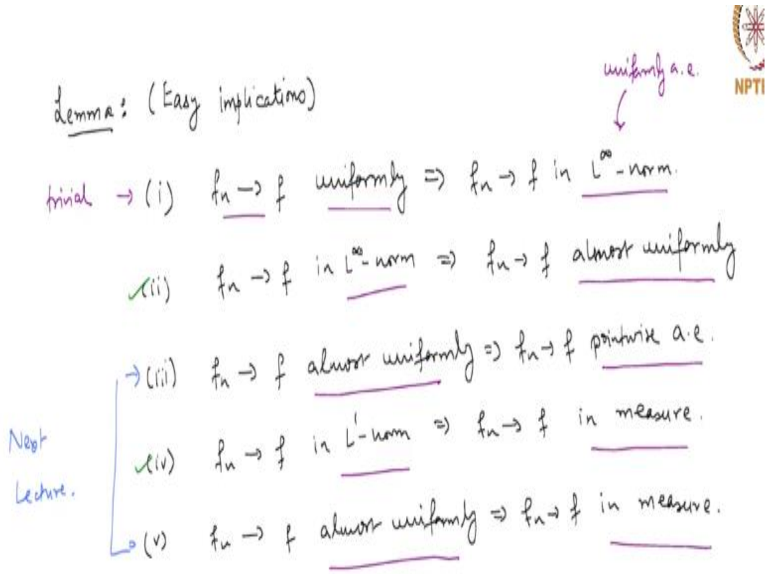

Now we can see some easy implication from that follow from the definitions of these most of convergence. So the first one is that if fn converges to f uniformly then it converges to f in L infinity norm. Remember that this was uniform convergence almost everywhere it was equivalent to uniform convergence almost everywhere. So this implication is actually trivial so this is trivial the second one is fn converges to f in L infinity norm implies fn converges to f is almost uniformly.

The third one is fn converges to f almost uniformly implies fn converges to f point wise almost everywhere. Fourth one is fn converges to f in L in 1 L1 norm implies f1 converges to f in measure. And the last one is fn converges to f almost uniformly then fn converges to f in measure. So the first one we have seen is X trivial this second I will only do the second and fourth in this lecture and I will leave the third and fifth for the next lecture.

So the second and fourth one are also quite easy so I will only do second and fourth and these 2 are will be taken up in the next lecture. So let me do the second and fourth one.

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Pf: (ii) To show: $f_n \to f$ in L^∞-norm $\Rightarrow$ $f_n \to f$ almost uniformly.

Convergence in L^∞-norm: Given $\epsilon > 0$, $\exists\, N \in \mathbb{N}$, s.t. $\forall n \geq N$.

$$\|f_n - f\|_{L^\infty} \leq \epsilon \iff \text{Convergence uniformly a.e.}$$

Convergence almost uniformly: Given $\epsilon > 0$, $\exists$ a set $E_\epsilon \in \mathcal{B}$, s.t. $\mu(E_\epsilon) \leq \epsilon$.

and $f_n\big|_{E^c} \to f\big|_{E^c}$ uniformly.

(i.e. $f_n \to f$ uniformly outside of E).

So for second one we need to show that fn converges to f in L infinity norm implies fn converges to f almost uniformly this was our second statement. So let us write down what it means for converges in L infinity norm this means that given epsilon. So this is to show given epsilon greater than 0 their exist a N such that for all n greater than or equal to N this L infinity norm of the difference of fn and f this is less than or equal to epsilon.

And so this is convergence in L infinity norm and convergence almost uniformly this is which says that given epsilon greater than 0. Their exist a set E so this set E depends on epsilon and it is measureable set in X such that the measure of E epsilon is less than or equal to epsilon and fn restricted to E compliment converges to f restricted to E complement uniformly. So we say that fn converges to f uniformly outside of E.

So this is convergence almost uniformly so if you have convergence in L infinity norm we have seen that this is the same as this is equivalent to convergence uniformly almost everywhere.

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So we have, given $\epsilon > 0$, $\exists\, N \in \mathbb{N}$, s.t. $\forall\, n \geq N$,

$$|f_n(x) - f(x)| \leq \epsilon \quad \text{for } x \text{ outside a } \mu\text{-null set } E_\epsilon.$$

$$\Leftrightarrow \quad f_n \to f \quad \text{uniformly outside } E_\epsilon,\ \mu(E_\epsilon) = 0 \leq \epsilon$$

$$\Rightarrow \quad f_n \to f \quad \text{almost uniformly.}$$

(iv) To show: $f_n \to f$ in L^1-norm $\Rightarrow f_n \to f$ in measure.

$$\|f_n - f\|_{L^1} \to 0 \text{ as } n \to \infty \Rightarrow \underbrace{\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\})}_{\to 0 \text{ as } n \to \infty}$$

So this means that so we have given epsilon greater than 0 their exist a N such that or all N greater than equal to N mode of fn x – fx less than or equal to epsilon for X outside a mu null set let me call it E. This is the same as saying that fn converges to f uniformly outside E and of course the measure of E is equal to 0 and of course this is less than or equal to epsilon. Notice that this mu null set E also depends on epsilon even that is null set it may change when you were a epsilon.

So now this implies that fn converges to f almost uniformly because your exceptional set is precisely this E epsilon that we have chosen here which is which is a mu null set. So this proves to for the fourth part we have to show that fn converges to f in L1 norm implies that fn converges to f in measure. So we have to show that when fn – f as L1 norm going to 0 as n goes to infinity then this implies that the measure of this set mode fn x – fx greater than equal to epsilon goes to 0 as n goes to infinity.

So this proof is quite easy because this kind of sets we know how to estimate and this is by the Markov's inequality.

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By Markov's inequality for $\epsilon > 0$, we have

$$\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) \leq \frac{1}{\epsilon} \cdot \|f_n - f\|_{L^1} \xrightarrow{n\to\infty} 0$$

fixed.

So by Markov's inequality for any positive epsilon we have that the measure of the set of points in x. Such that fn x – fx as modulus greater than epsilon is bounded above by 1 over epsilon times the L1 norm of fn – f. So this was the statement of Markov's inequality and because epsilon is fixed this is fixed this goes to 0 as n goes to infinity by our hypothesis. So L1 convergence implies convergence in measure.