Measure Theory
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Module No # 11
Lecture No # 51
Various modes of convergence of measureable functions

So until now we have seen the concept of Lebesgue integrals and we have emphasized particular properties of absolutely integrable or L1 functions. And in this lecture we will come to a new topic which is that of various modes of convergence in which a sequence of real or complex measurable functions can converge to another measureable functions on a measured space.

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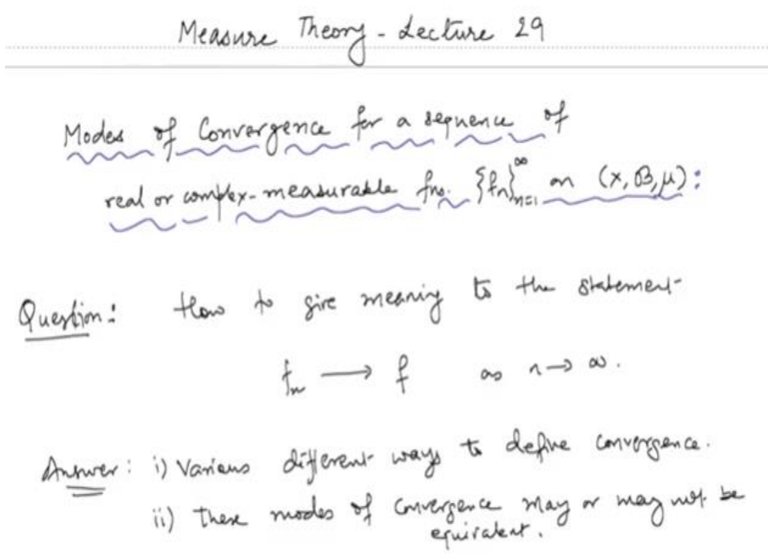

So the idea here is to answer the following question is how to give meaning to their statement that f n convergence to f as n goes to infinity. So as we will see the answer is that there are various ways to do this various different ways to define convergence. So this is first part and the second part is that this ways different ways or modes we have also called it modes. So these modes of convergence may or may not be equivalent.

So in general we will have various different ways of defining convergence for a sequence of real or complex measureable functions. And this modes of convergence may or may not be equivalent meaning that each may be in equivalent to the other. So we will see various examples of such things but first let me recall what we already know from our undergraduate days.

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i) Pointwise Convergence: $f_n \to f$ pointwise on X

if and only if for any $\epsilon > 0$, and $x \in X$, $\exists N \in \mathbb{N}$

s.t. $|f_n(x) - f(x)| \leq \epsilon \quad \forall n \geq \underbrace{N}_{\text{depends on the point } x \in X}$.

ii) Uniform Convergence: $f_n \to f$ uniformly on X

if and only if for any $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$|f_n(x) - f(x)| \leq \epsilon \quad \forall n \geq \underbrace{N}_{\text{does not depend on } x}$ and $\forall x \in X$.

So 2 basic notions of convergence so the first one is point wise convergence we have already seen point wise convergence. This is the most basic way of convergence for a sequence of functions and it says that f n convergence to f point wise on X if and only if for any epsilon greater than 0 and x in X. There exist a natural number N such that mode of fn x – fx is less than or equal to epsilon for all n greater than equal to N.

So of course this N here this N depends on the point X so once you are chosen a point the threshold N after which is valid this inequality is valid depends on the chosen point x. Now for uniform convergence this is another basic notion of convergence uniform convergence fn convergence to uniformly on X if and only if for any epsilon greater than 0. Their exist N such that mode of fn x – fx is less than or equal to epsilon for all n greater than equal to n and x in X.

So this is for all x in X so this the difference between uniform and point wise convergence is of course that this N now does not depend on x on this chosen point x depend on x. And this N the threshold after, which inequality holds can be chosen uniformly over or points x in your space X.

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example: $f_n : \mathbb{R} \to \mathbb{R}$.

$f_n(x) = \frac{x}{n}$, $n \geq 1$.

easy to check: $f_n \to f \equiv 0$ pointwise but not uniformly. (but does converge to zero locally uniformly).

(iii) Pointwise convergence almost everywhere:

$f_n \to f$ pointwise a.e. $\Leftrightarrow$ $f_n(x) \to f(x)$ for μ-a.e. $x \in X$.

(i.e $f_n \to f$ pointwise outside of a μnull-set).

So an example of a function which converges of a sequence of functions converges point wise but not uniformly is the following. So if you take fn x so fn let me take the real line and our functions are real valued measureable and we can define fn x to be x over n. So this is for n greater than equal to 1 and so this is the sequence of functions. So easy to check that fn converges to the function 0 point wise but not uniformly. We have seen that even if it is not uniform it can be locally uniform and in this case we do have local uniform convergence.

So but does converge to 0 locally uniform so meaning that it converges uniformly to 0 on each bounded subset of R. So this is a standard example of sequence of functions which converges point wise to a certain function but does not converge uniformly. On the other hand we have already seen point wise convergence almost everywhere. So we say that fn converges to f point wise almost everywhere if and only if fn x converges to fx for mu almost every x in X.

Meaning that outside of a null set which means that fn converges to f point wise outside of a null set or a mu null set meaning that the measure of the points on which fn x may not converge to fx as mu measure 0. So of course point wise converges implies point wise convergence almost everywhere this is trivially true. And of course uniform convergence almost implies point wise convergence.

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Sequence of implications:

$$\text{Uniform Conv.} \;\underset{\not\Leftarrow}{\Rightarrow}\; \text{pointwise convergence} \;\underset{\not\Leftarrow}{\Rightarrow}\; \text{pointwise conv. a.e.}$$

e.g. $f_n(x) = \begin{cases} \frac{x}{n} & x \in \mathbb{R}\setminus\{1\} \\ (-1)^n & x = 1. \end{cases}$

iv) <u>Convergence in L^1-norm:</u> $f_n \to f$ in L^1-norm.

$\iff \|f_n - f\|_{L^1} \to 0$ as $n \to \infty$.

So we have sequence of implications for these 3 modes of convergence so this is that uniform convergence implies point wise convergence and point wise convergence implies point wise convergence almost everywhere. We have already seen that we cannot have the reverse implication here, meaning that point wise convergence does not imply uniform convergence and it is not also quite easy to see that, we also do not have reverse implication here by just modifying a sequence of function that convergence point wise everywhere.

On a set of measures 0 we can easily construct point wise convergence almost everywhere which does not have point wise convergence everywhere. For example so for just for this one we can take this sequence of function on r x over n f and x is x over n x in R. And we can leave out just 1 point let us say the point 1 and we can take the sequence -1 to the power n if x = 1. So meaning that fn x is the same function x over n if x is not 1 and it is this sequence -1 to the power n if x = 1.

And of course when x = 1 there is sequence -1 to the power n does not converge to 0 in fact it does not converge to anything. And if x is not equal to 1 then it converges to 0 so outside of single point which has measure 0 we see that we have point wise convergence but it does not converge anything on this set of measure 0 which is the point x = 1. Now we have also seen another mode of convergence which is convergence in L1 norm.

So we say that fn converges to f in L1 norm if and only if this difference of fn – f the L1 norm of the difference goes to 0 as n goes to infinity. Now if you want to compare convergence in L1 norm with uniform convergence or point wise convergence or point wise everywhere convergence then neither does each of these imply L1 convergence or L1 convergence implies each of these uniform point wise or point wise almost everywhere convergence. So we have already seen for example the typewriter sequence so let me make it a remark here.

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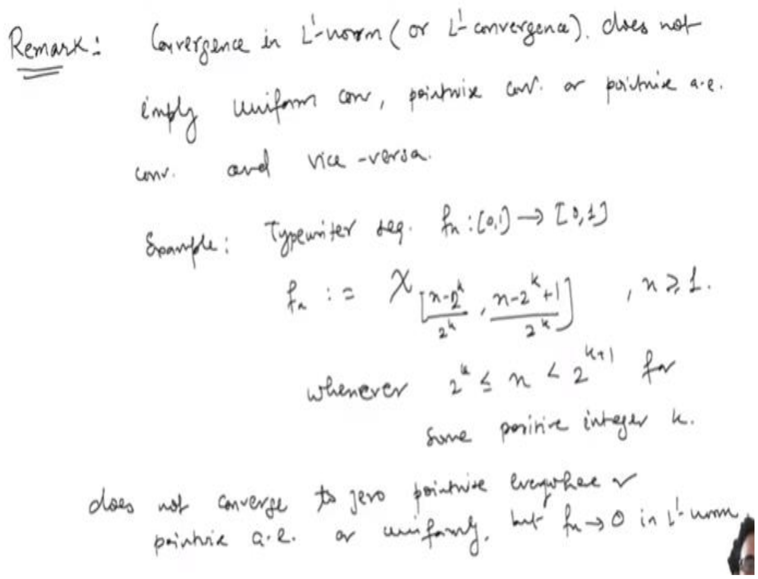

Convergence in L norm we will also call it sometimes L1 convergence does not imply uniform convergence point wise convergence or point wise almost everywhere convergence and vice versa. Meaning that all this 3 may not imply L1 convergence so, we have seen the example of typewriter sequence. So this was the sequence on function on the real line again which was given by fn was the indicative function of the set 2 to the power n – 2 to the power k over 2 to the power k n – 2 to the power k + 1 over 2 to the power k.

Whenever 2 to the power k is less than equal to n is less than 2 to the power k + 1 for some positive integer k. So this is for n greater than or equal to 1. So this sequence as we have seen actually I should we can also take 0, 1 here 0, 1 to 0, 1 these are all subsets of the interval 0, 1 and it only takes value is 0 and 1. So we can restrict to this restrict our functions fn to this interval 0, 1 and taking values in 0, 1.

So we have seen that this functions fn merge across this intervals 0, 1 back and forth and so it does not converge does not converge to 0 point wise everywhere or point wise almost everywhere or uniformly. But fn converges to 0 in L1 norm this is because the support of this functions fn have smaller and smaller measure as n goes to infinity and so it goes to 0 in L1 norm. But it does not converge to 0 point wise everywhere or point wise almost everywhere or even uniformly.

So this is the standard counter example for sequence of functions so it converges in L1 norm but does not converge in either of these 3 standard ways of convergence. It is also quite easy to come up with examples in which you have point wise convergence but not in L1 norm as well.

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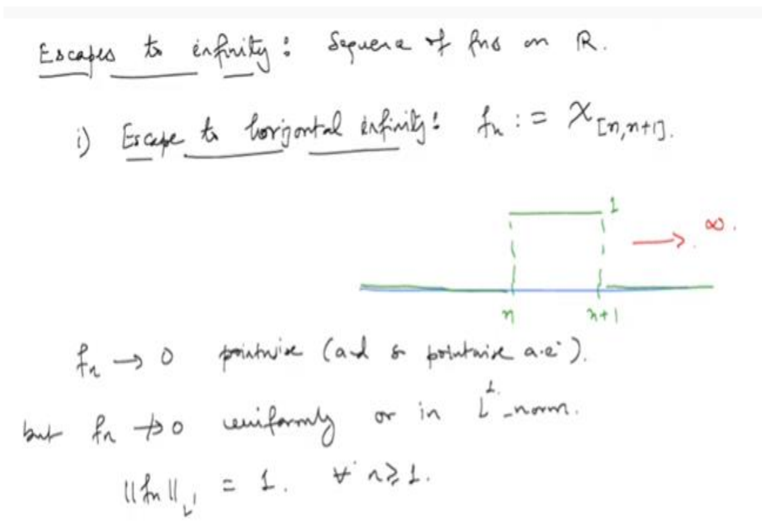

Now before we other modes of convergence it is a very useful to keep in mind some class of examples which violate 1 or the other modes of convergence and these are called escapes to infinity. So the first one is escape to horizontal infinity so all of these are examples on the real line. So these are sequence of functions on the real line so the first one is escape to horizontal infinity here we take the function sequence of functions defined by the characteristic function of the interval n, n + 1.

So this is the sort of moving bump if you want so here is your point n here is n + 1 and fn is 1 at these points within these intervals and it is 0 elsewhere. And as n increases this goes to infinity so it is like a bump moving horizontally to infinity. And we can say what kind of convergence fn satisfies so first of all fn converges to 0 point wise and therefore point wise almost everywhere. But fn does not converge to 0 uniformly or in L1 norm because the functions are 1 at some interval.

So the uniform convergence is invalidated and the L1 norm of this function fn is simply 1 for all n therefore it does not go to 0 in L1 norm. So we see that fn goes to 0 only in point wise and point wise almost everywhere but it does not go to 0 in uniform convergence or in L1 norm.

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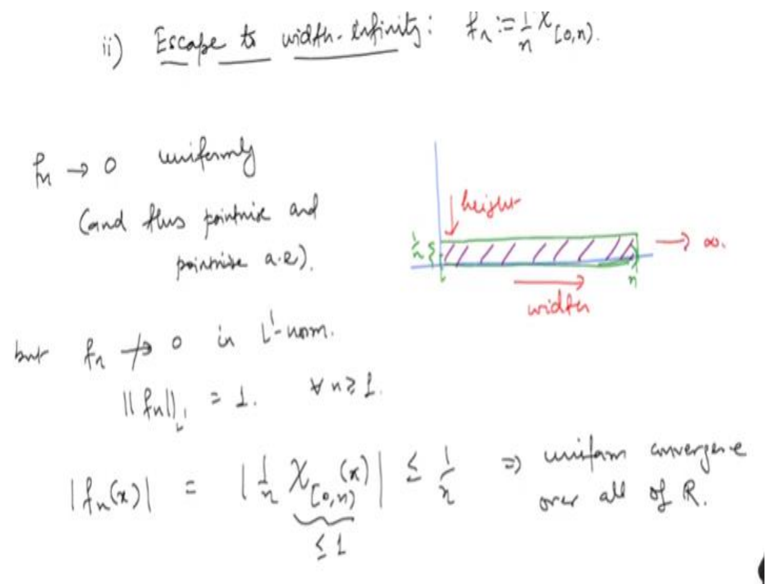

Another escape to width infinity in this case to be the f1 to the sequence of functions 1 over n Chi of 0, n so indicated function of the interval 0, n . So here you have this interval 0 to n and the height of the function is 1 over n so this is 1 over n. And so we have that it cannot converge to 0 in L1 norm but it does converge to 0 point wise and in fact it converges to 0 uniformly. So fn converges to 0 uniformly and thus point wise and point wise almost everywhere.

But fn does not converge to 0 in L1 norm because again the L1 norm is 1 throughout the sequence so it does not converge the 0 in L1 norm. But if you want to bound the modulus of fn x this is simply the modulus of Chi 1 over n Chi 0, n x and because this is less than equal to 1 this whole thing is less than or equal to 1 over n. And so this implies uniform convergence over all of r. So we see that it converges uniformly but not in L1.

So this is the escape to width infinity because the height goes to 0 but the width was to infinity this goes to infinity.

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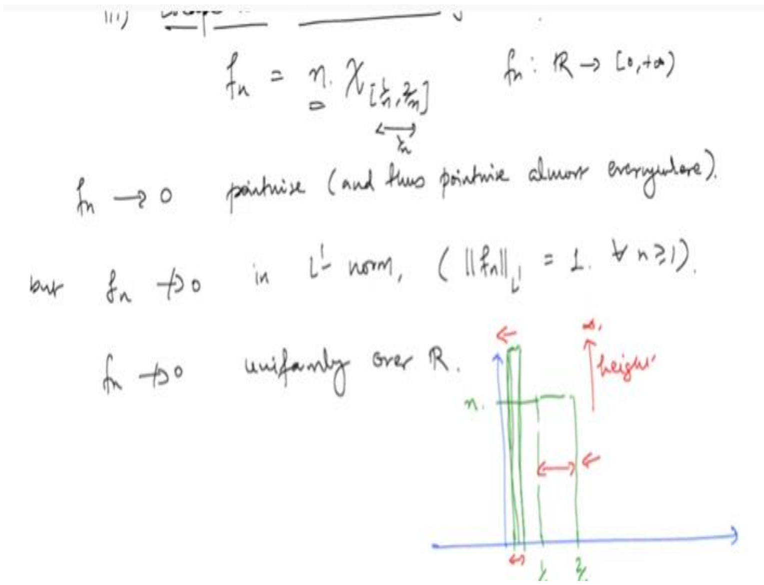

Another mode of convergence is escape to vertical infinity in this example we take fn to be n times the indicative function of the interval 1 over n to 2 over n. So again this is a functions are defined on r with values in the positive real’s and we see that fn converges to 0 point wise and therefore point wise almost everywhere. But fn does not converge to 0 in L1 norm this is because the norm of fn L1 norm of fn for each n this is equal to 1 because the width of this interval is 1 by n and you are multiplying by n.

So the L1 norm is just the integral or the area under the curve if you want and this is 1 for all n greater than or equal to 1. So it does not converge to 0 in L1 norm but also fn does not converge to 0 uniformly over r this is quite easy to see as these values n become larger and larger. So you cannot have a uniform bound over all of r so this is called escape to vertical infinity because we have these fn’s are of this sort where you have 1 over n 2 over n and the value is this value is n.

And so the height vertical height goes to plus infinity while this pump moves towards smaller if the weight becomes smaller and smaller. So this length becomes smaller and smaller as n increases so for example for higher n it could be like this and it goes towards 0. So even though the width decreases the height goes to plus infinity which causes problems in the L1 norm and uniform convergences.

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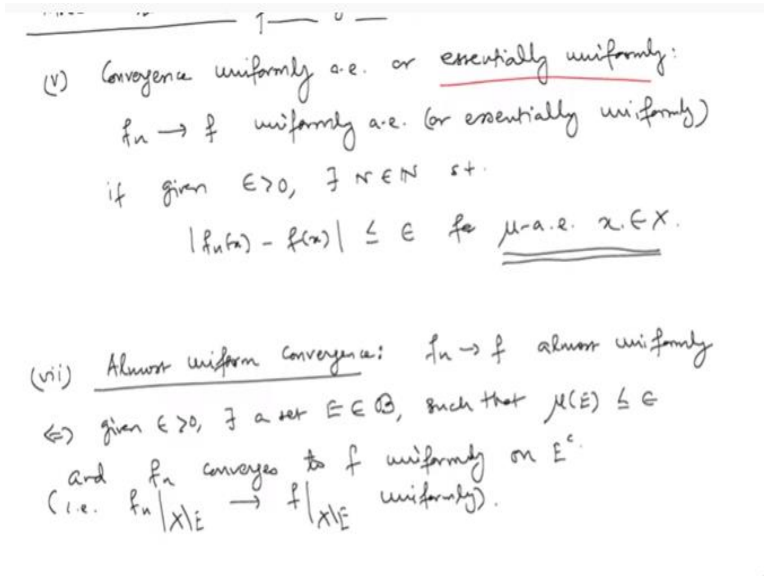

Now we will define 3 more modes of convergence so the first one rather this is the fifth mode of convergence we have already seen 4. So this is the fifth one this is called convergence uniformly almost everywhere or essentially uniformly. So we say that fn converges to f uniformly almost everywhere or essentially uniformly. If given epsilon greater than 0 their exist an N such that mode of fn x – fx is less than or equal to epsilon for mu almost everywhere x in X.

So this condition for uniform convergence is now replaced for by the condition that it only holds for x outside of a null set in X. So this is uniform convergence almost everywhere or what is called essentially uniformly convergence essentially uniformly. So this is the first one in our extra mode of convergence this is the second one is as follows this is called almost uniform convergence.

So the terminologies are quite similar but the behavior is quite different so one has to be very careful when considering this terms and one as to be sure of what the definition of this terms are. So almost uniform convergence means that fn converges to f almost uniformly if and only if given epsilon greater than 0 their exist a measureable set E which belongs to the sigma algebra B this is sigma algebra B.

Such that the measure of E is less than or equal to epsilon and fn converges to f uniformly on E complement. Meaning that the restriction of fn to E complement converges to the restriction of, f to E complement uniformly.

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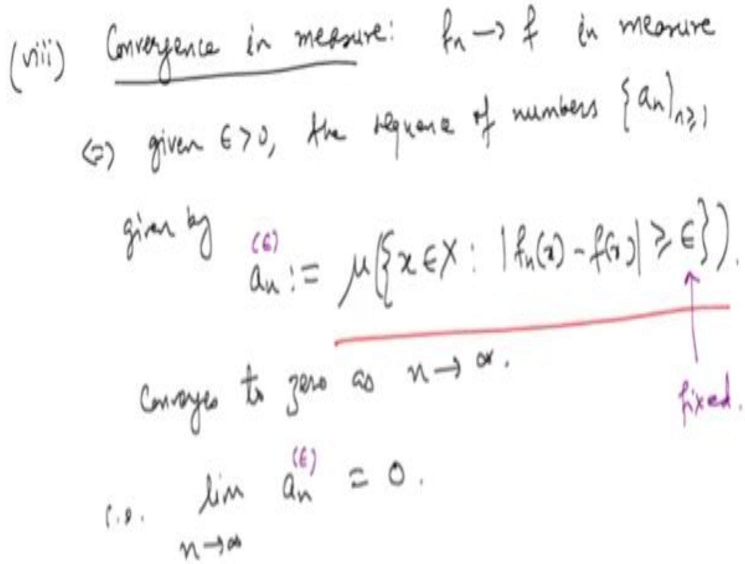

So this is almost uniform convergence and the third one is convergence in measure so fn converges to f in measure if and only if given epsilon greater than 0 and the sequence of numbers let us call it an given by. So this nth term in the sequence is given by the measure of points in x such that fn x – fx the modulus is greater than or equal to epsilon converges to 0 as n goes to infinity meaning that the limit as n goes to infinity an this is equal to 0 where an is given by this formula.

So here note that this epsilon in this sequence this epsilon is fixed and so once you fixed epsilon you get a sequence which depend so this number epsilon so here also this sequence depends on this number epsilon but as n tends to infinity this sequence to infinity. And this should happen for any epsilon greater than 0. So for different values of epsilon you get different sequences yet all this sequences have the limit 0 so this is called convergence is measure.