Measure Theory
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Lecture - 50
L 1 Functions on R d: the Riesz-Fischer Theorem

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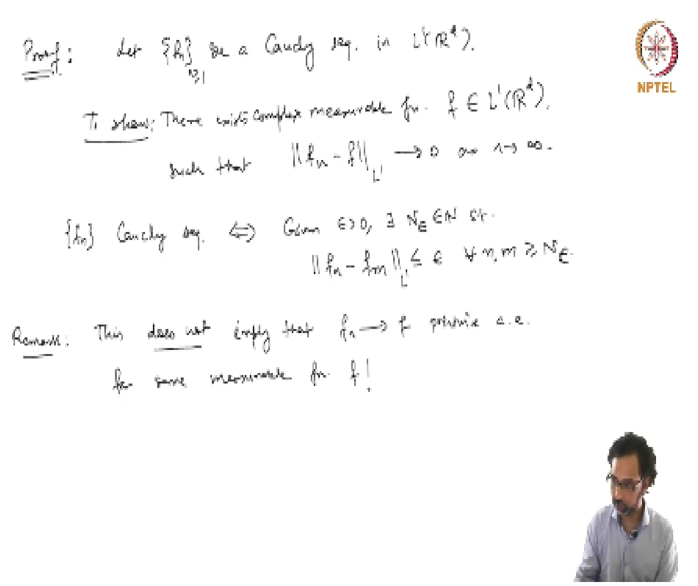

So, let us see the proof of the Riesz-Fischer theorem. So, let f n be a Cauchy sequence in L 1 R d. So, here again I am writing f n rather than the equivalence classes because the meaning is quite clear. We only have equality almost everywhere for the other representatives of the same class. So, rather than writing box f n; we just writing f n as any representative in the equivalence class.

So, if it is a Cauchy sequence in L 1. So, we have to show that there exists a complex measurable function f which belongs to L 1 R d such that norm of f n - f in the L 1 0 goes to 0 as n goes to infinity. So, f n Cauchy is a Cauchy sequence, is the same as saying that for all given epsilon greater than 0. There exists a number n epsilon such that the norm of f n - f m is less than or equal to epsilon for all n and m greater than or equal to this number and epsilon.

So, this is what a Cauchy sequence means in our case. And let me remark here that this does not imply that f n converges to f point wise almost everywhere for some measurable function f. So, L 1 convergence which means that if you have a Cauchy sequence in the L 1 norm, it does not imply that you have point wise everywhere almost everywhere convergence.

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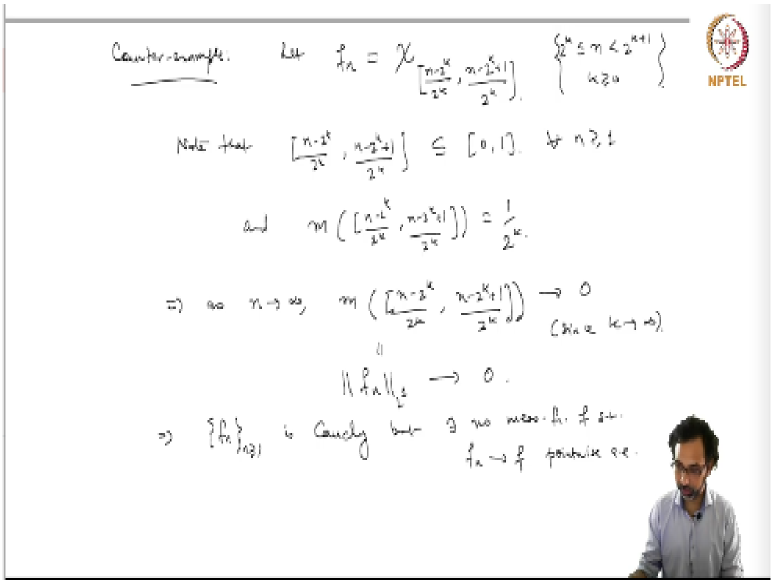

So, let me give a counter example to this fact. So, let f n be the indicator function of the following set. So, for n greater than or equal to 2 to the power k less than 2 to the power k + 1 for k greater than 0. We take n - 2 to the power k over 2 to the power k and n - 2 to the power k + 1 over 2 to the power k. So, we take the indicator function for this interval and we know note that these intervals n - 2 to the power k + 1 by 2 to the power k.

These are all subsets of 0 1. Okay. So, for all n greater than or equal to 1 where k is chosen such that n lies between 2 to the power k and 2 to the power k + 1. So, first of all, all of these sets are within 0 1 and the measure of these sets, these intervals over 2 to the power k. This is simply 1 over 2 to the power k. So, as n goes to infinity the measure of these sets n - 2 to the power k by 2 to the power k, n - 2 to the power k + 1 over 2 to the power k goes to 0; since k also goes to infinity if n goes to infinity.

And note that this measure is nothing but the L 1 norm of f n. So, this means that the L 1 norm of f n goes to 0 because this is a simple function. So, the L 1 norm is simply the measure of this interval. So, the L 1 norm goes to 0 and this implies that we have a Cauchy sequence. This sequence f n; n greater than or equal to 1 is Cauchy but there exists no function no measurable function f says that f n converges to f point wise almost everywhere.

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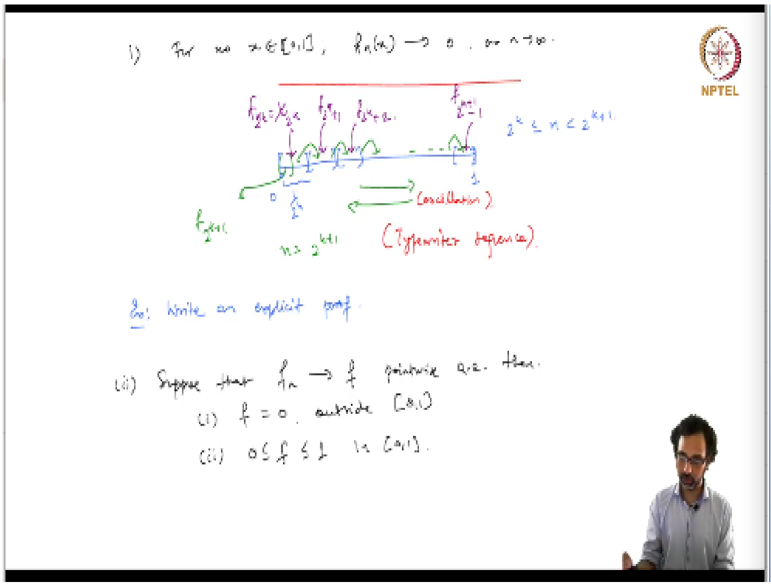

So, let us see why this is the case. So, first is that for no x in 0 1, f n x converges to 0 as n goes to infinity. This is because the nature of this sequence of indicator functions is such. So, if you take the interval 0 1 and 2 to the power k is less than or equal to n less than 2 to the power k + 1. Then it is then this interval is subdivided into subintervals of length 1 over 2 to the power k.

So, like this and then when n is 2 to the power k, this is. So, let me write it here. So, this is chi 2 to the power k. This is f 2 to the power k and then f 2 to the power k + 1 then f 2 to the power k + 2 and so on. And finally, when you have sorry here 2 + 1 + 2 and so on and lastly the last subinterval is f 2 to the power k + 1 - 1. So, you see that when n is between 2 to the power k and 2 to the power k + 1 in the exponent.

We have that the indicator functions moves from 1 to the next of these subintervals. And then when n = 2 to the power k + 1; if n = 2 to the power k + 1 then it goes back here with a smaller interval and right at the beginning of the interval 0 1. So, here this is f 2 to the power k + 1. So, the indicator functions keeps moving from back and forth in the interval 0 1 and this is why it is called the Typewriter sequence.

I do not know if you have seen an old Typewriter where you had to move back the lever to print again. So, it is this why is called the Typewriter sequence and this is why because of this oscillation back and forth, this is the oscillation of these indicator functions back and forth. We do not have point wise convergence. So, this is just a heuristic argument but you can always write down an explicit proof.

So, I will leave it as an exercise write an explicit proof for this fact that f and x does not converge to 0 for any x. Now, the second one is that suppose that f n converges to some f point wise almost everywhere. Then first of all that f = 0 outside 0 1 because f n is 0 outside 0 1 and secondly f lies between 0 and 1 in within the interval 0 1. So, this is just from the nature of the sequence f n's.

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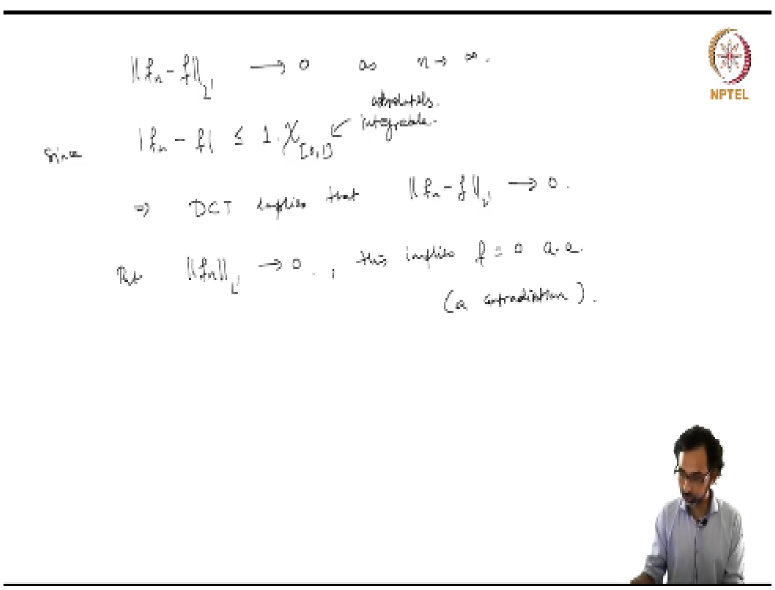

Then we see that the norm of f n - f the L 1 norm is going to 0 as n tends to infinity because since f n – f. Now, these are both between 0 and 1. So, therefore this is bounded above by 1. And it is 0 outside the interval 0 1. So, it is bounded by the function, the indicator function of 0 1 and this is an integral function; this is integral absolutely integral L 1 function. And so, this implies by the dominated convergence theorem that implies that the norm of f n - f L 1 norm goes to 0 but we have already seen that the norm of f n goes to 0.

And since we are in a matrix space. It is always **(( )) (11:29)** off. So, we cannot have 2 limits; this implies that f = 0 almost everywhere which is a contradiction from the previous part. So, we see that even if you have a Cauchy sequence in L 1 norm, it does not imply that it converges point wise everywhere almost everywhere to a function f. So, now that we have seen this counter example let us go back to our proof of the Riesz-Fischer theorem.

So, we have a Cauchy sequence in the L 1 norm. Now, even though we do not have point wise everywhere almost everywhere convergence. We still can extract a subsequence which converges point wise almost everywhere. So, this is the idea for the proof.

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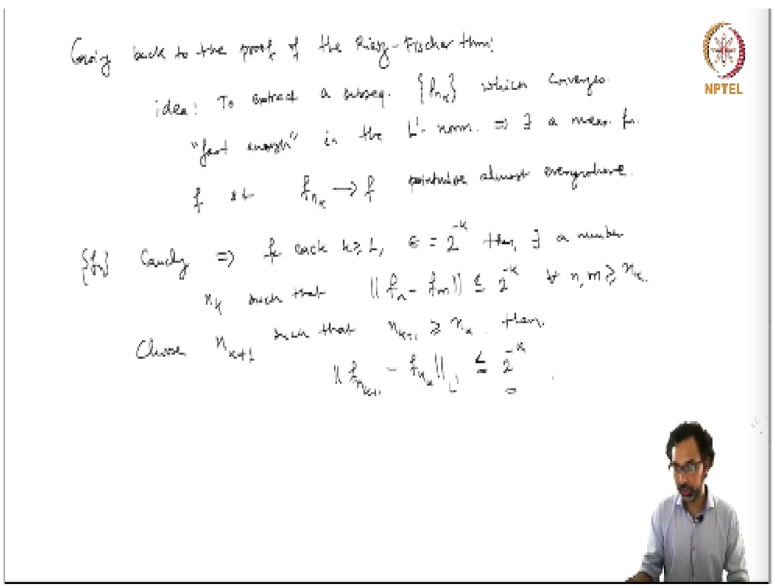

So, going back to the proof of the Riesz-Fischer theorem. So, the idea is to extract a subsequence f n k of the sequence f n which converges fast enough and what do I mean by fast enough I will make clear in a while. So, which converges fast enough in the L 1 norm. So, this would imply that there exists a measurable function f states that f n k converges to f point wise almost everywhere.

So, this is the idea. And now let us see the details for this idea. So, of course f n Cauchy is Cauchy implies that for each k greater than or equal to 1 if we take epsilon to be 2 to the power – k. Then there exists a number small n k such that the norm of f n - f m is less than or equal to 2 to the power - k for all n and m greater than or equal to this number n k. So, if we choose n k + 1 such that n k + 1 is greater than or equal to n k.

Then we have f n + n k + 1 - f n k. The L 1 norm is bounded over by two to the power - k and this is the subsequence that we want and it converges fast enough because now we have on the right hand side and absolutely sum-able sequence.

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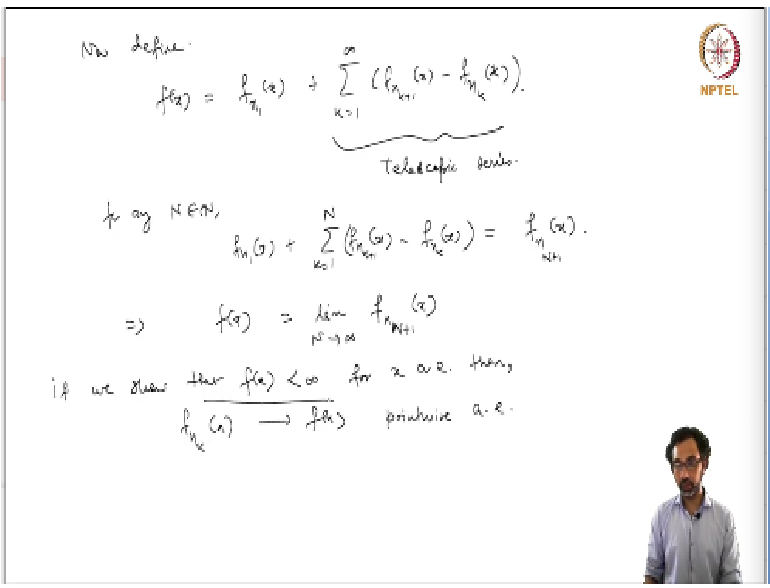

So, now define f x to be the sum f n 1 x + k from 1 to infinity f n k + 1 x - f n k x, f n k. Now, this is a Telescopic series and if you look at the partial sums for this series. So, for any capital N in N if we let, if you take f n 1 x + k = 1 to capital N f n k + 1 x - f n k x. Then this is nothing but simply f n capital N + 1 of x. So, which means that f x is the limit as capital N goes to infinity of this partial sum f n n + 1 x.

And so, if we prove that this limit exists almost everywhere then f will be a measurable function which for which this subsequence convergence always almost everywhere to f. So, if we show that f x is finite for x almost everywhere. Then this subsequence f n k converges to f x for almost every x. So, this is a point wise convergence almost everywhere. So, we have to show that this is a finite for almost every x.

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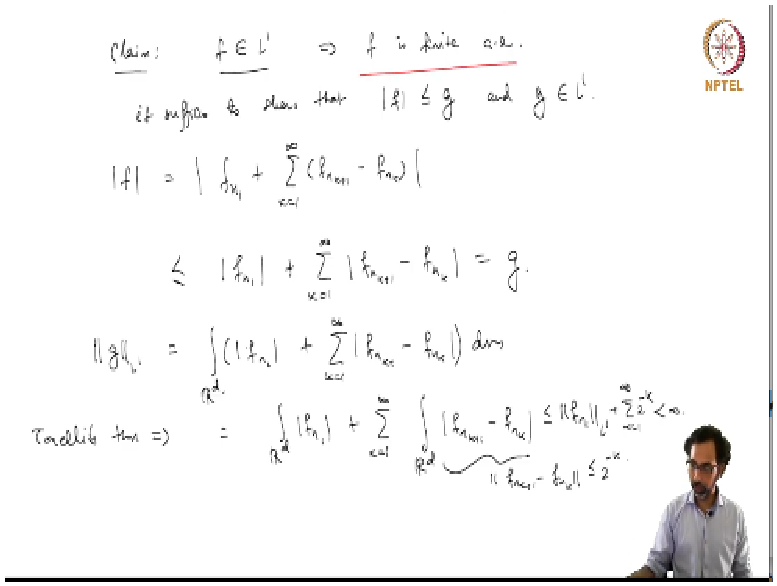

So, we claim that f is in L 1 function. So, this would imply that f is finite almost everywhere because we have already seen that L 1 functions are finite almost everywhere. So, to show this, it suffices to show that the modulus of f is less than or equal to g and g belongs to L 1. So, first of all the modulus of f is less than or equal to g by construction. So, mod f is equal to the modulus of f n 1 plus the sum f n k + 1 - f n k, k = 1 to infinity.

And then you can use limit on the triangle inequality. So, you will get modulus of f n 1 plus the sum k = 1 to infinity mod f n k + 1 - f n k. And this is nothing but g k. This is the function g. Sorry. And so, modulus of f is bounded above by g. Now; to show that g is L 1. We consider the L 1 norm of g which is equal to the integral of this function modulus of f n 1 plus this sum k = 1 to infinity modulus f n k + 1 - f n k d m. So, this is over R d.

And we have in the integral of this function which is a non-negative function and now we can use Tonelli's theorem to interchange the sum and the integral. So, this implies that this is equal to the integral R d f n 1 plus the sum k = 1 to infinity of the integrals over R d f n k + 1 - f n k. And this is nothing but the L 1 norm of f n k + 1 - f n k and each of them is less than or equal to 2 to the power - k.

So, therefore this is less than or equal to the L 1 norm of f n 1 plus the sum k = 1 to infinity to the - k and this is an absolutely sum-able series. So, it is finite. So, the L 1 norm of g is finite which means that f is an L 1 function which means that f is finite almost everywhere. So, we have shown that f is finite almost everywhere.

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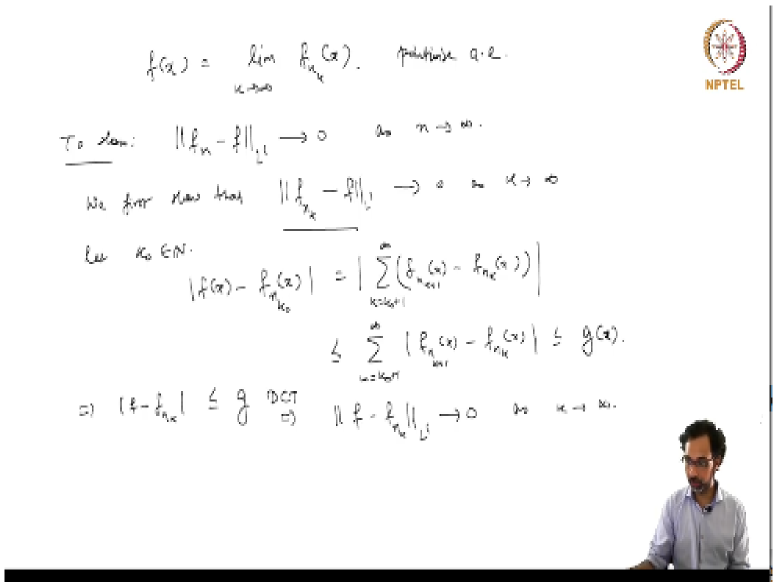

And so, f x is the limit as k tends to infinity of this f n case. This is point wise almost everywhere. So, this subsequence converges to a measurable function f. Now, we have to show that the whole sequence f n converges in L 1 norm to f. So, to show that the norm of f n - f L 1 goes to 0 as n goes to infinity. We first show that the subsequence f n k also converges to f in the L 1 norm as k goes to infinity.

So, this is n k f of n k. How do we show this? So, let us fix k 0 then if you take f x - f n k 0 f of x. Then this is nothing but modulus of the sum k = k 0 + 1 to infinity f n k + 1 x - f n k x and this is of course less than or equal to the sum of all this modulus of all these individual terms f n k + 1 x - f n k x and this is less than or equal to the function g.

So, we see that f - f n k is dominated by this function g and this implies by the DCT; the dominated convergence theorem shows that the norm of f - f n k goes to 0 as k goes to infinity. So, we have shown that the subsequence converges to the function f in the L 1 norm.

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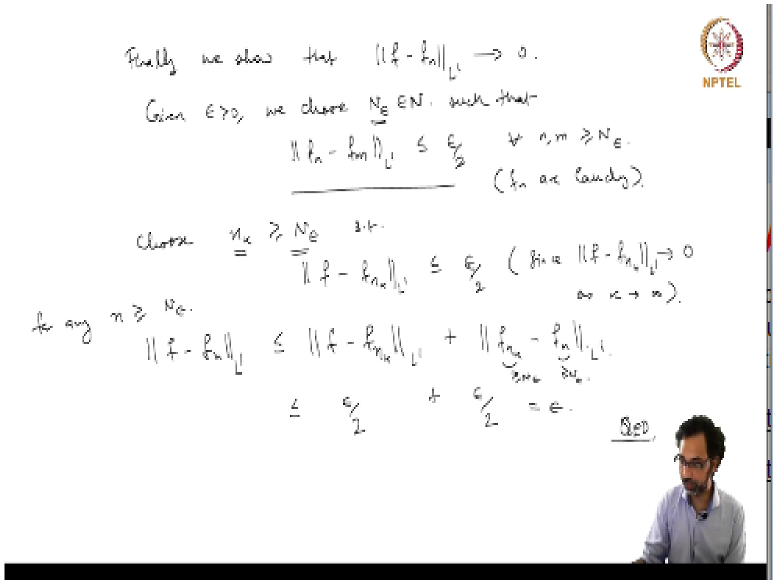

Now, final step, finally we show that the norm of f - f n L 1 goes to 0. So, given epsilon greater than 0. We choose a number n epsilon says that the norm of f n - f m is less than equal to epsilon by 2 for all n and m greater than or equal to n epsilon. This is by the Cauchy property of the sequence f n. So, this is by f n Cauchy. And now choose n k greater than or equal to n epsilon such that the norm of f - f n k L 1 is also less than or equal to epsilon by 2.

And this, we can do since the norm of f - f n k goes to 0 as k goes to infinity. So, we can choose our n k greater than or equal to this number n epsilon that we have chosen here and we get a similar inequality. So, finally if you take the norm of f – f n. So, for any n greater than or equal to n epsilon. We have that f - f n is less than or equal to the norm is less than or equal to f - f n k. So, this is the chosen n k L 1 + f n k - f n L 1.

And this is less than or equal to epsilon by 2 and also this second term is less than or equal to epsilon by 2 because both these terms are greater than or equal to n epsilon. Both these indices like greater than or equal to n epsilon for which this holds by the Cauchy criteria. So, this is epsilon. Finally, we are done and we shown, we have shown that f n converges to f in the L 1 norm. So, as a corollary of this proof so, this finishes the proof of the Riesz-Fischer theorem.

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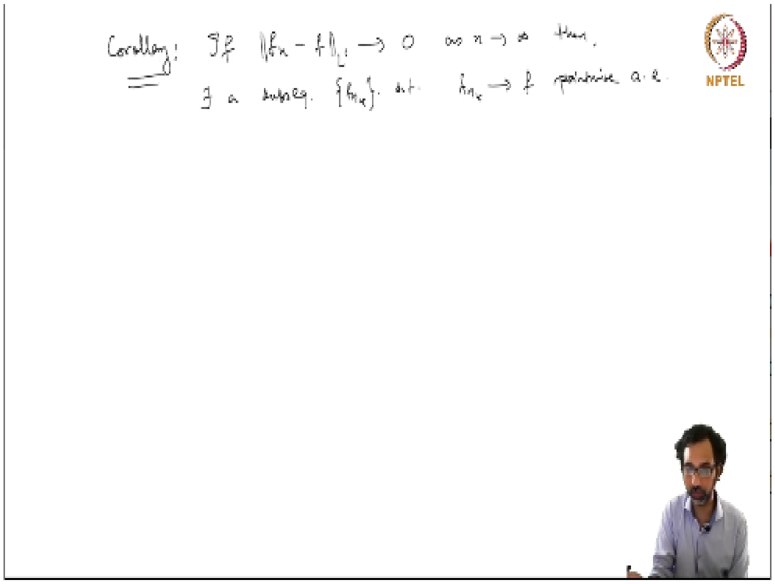

But as a corollary, we also obtain the important result that if f n - f the L 1 norm goes to 0 as n goes to infinity. Then there exists a subsequence f n k such that f n k converges to f point wise almost everywhere. So, we can choose our f n k such that this is a point wise almost everywhere convergence and this is always true whether or not f n converges to f point wise almost everywhere.