MEASURE THEORY

PROF. INDRAVA ROY

Lecture-5
Elementary Sets and Elementary Measure - Part 2

For general $n \geq 1$:

$$E = \bigcup_{i=1}^{N} B_i$$

$$B_i = \underbrace{I_{1,i}} \times I_{2,i} \times I_{3,i} \cdots \times I_{n,i}$$

Consider the union of intervals $E_1 \subseteq \mathbb{R}$

$$\begin{array}{ccc} I_{1,1}, & \underbrace{I_{1,2}}, \ldots & , I_{1,N} \\ \uparrow & & \uparrow \\ B_1 & B_2 & B_N \end{array}$$

E_1 is an elementary subset of $\mathbb{R}$.

$\Rightarrow \quad E_1 = \bigcup_{k=1}^{N_1} J_{1,k}$ where $J_{1,k}$'s are disjoint intervals.

Similarly, $\quad E_j = \bigcup_{k=1}^{N} I_{j,k} = \bigcup_{k=1}^{N_j} J_{j,k}$

Now for general n, we can easily generalize our argument by using our case for $n = 1$. So for general $n \geq 1$, we have E is a finite union of boxes B_i, i.e.,

$$E = \bigcup_{i=1}^{N} B_i,$$

where each box B_i is of course a Cartesian product of intervals, i.e.,

$$B_i = I_{1,i} \times I_{2,i} \times \cdots \times I_{n,i}.$$

So there are n intervals because we are in dimension n and each B_i is a Cartesian product of intervals.

Department of Mathematics, The Institute of Mathematical Science, Chennai.

Note that $I_{1,i}$ is the first interval of the Cartesian product of B_j for each $j = 1, \cdots, N$. Now we consider the union of these intervals

$$I_{1,1},\ I_{1,2},\ \cdots,I_{1,N}$$

denoted by $E_1 \subseteq \mathbb{R}$, i.e.,

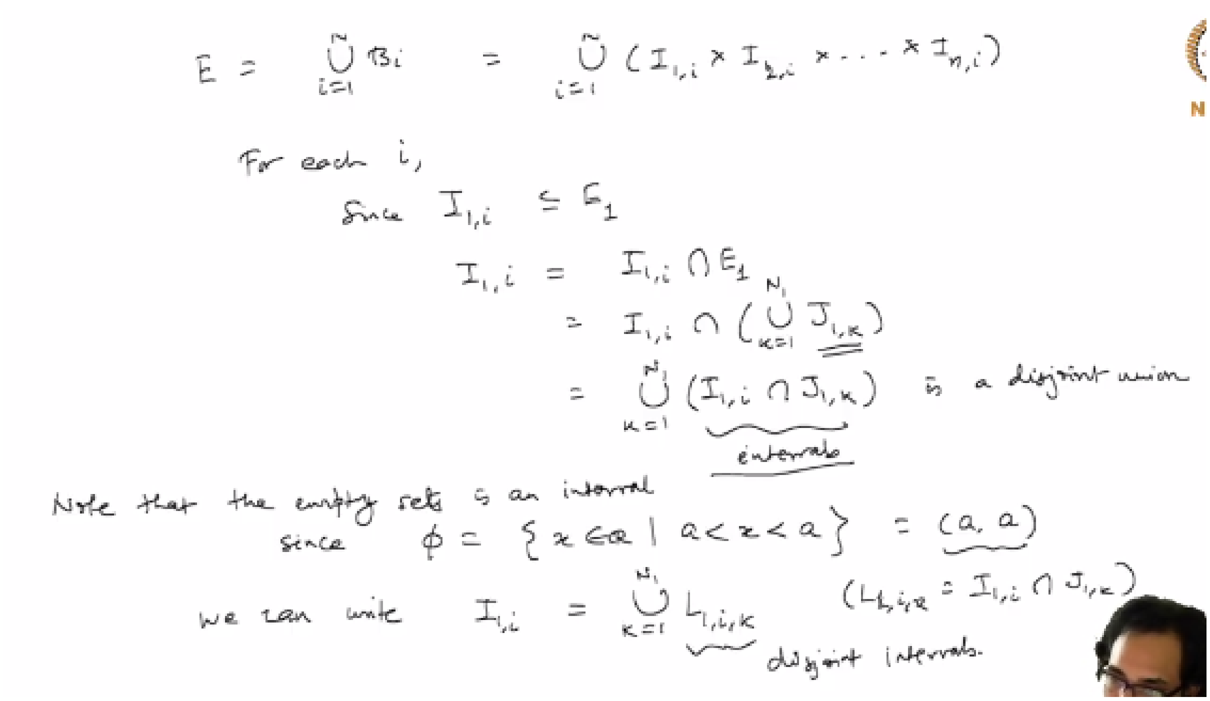
$$E_1 = \bigcup_{i=1}^{N} I_{1,i}.$$

Now E_1 is an elementary set. So write

$$E_1 = \bigsqcup_{k=1}^{N_1} J_{1,k},$$

where $J_{1,k}$'s are disjoint intervals. Similarly, we do this for

$$E_j = \bigcup_{i=1}^{N} I_{j,i} = \bigsqcup_{k=1}^{N_j} J_{j,k}.$$



Now I am going to write this Cartesian product. Write

$$\begin{aligned} E &= \bigcup_{i=1}^{N} B_i \\ &= \bigcup_{i=1}^{N} I_{1,i} \times I_{2,i} \times \cdots \times I_{n,i}. \end{aligned}$$

Note that for each i,

$$I_{1,i} \subseteq E_1.$$

So

$$\begin{aligned} I_{1,i} =& I_{1,i} \cap E_1 \quad (\text{since } I_{1,i} \subseteq E_1) \\ =& I_{1,i} \cap \Big(\bigsqcup_{k=1}^{N_1} J_{1,k}\Big) \\ =& \bigsqcup_{k=1}^{N_1} (I_{1,i} \cap J_{1,k}). \end{aligned}$$

Note that $I_{1,i} \cap J_{1,k}$ are intersection of intervals, so they can be either empty or it can be a non-empty interval, in both cases it is an interval. Note that the empty set is an interval from our conventions, since you can take the empty set as

$$\phi = \{x \in \mathbb{R} : a < x < a\} = (a, a).$$

So in this union all of them intervals and they can be either empty or non-empty, but nevertheless this is a disjoint union because each this $J_{1,k}$'s are disjoints for varying of k. So we can write

$$I_{1,i} = \bigsqcup_{k=1}^{N_1} L_{1,i,k},$$

where $L_{1,i,k} = I_{1,i} \cap J_{1,k}$ and these are all disjoint intervals in $\mathbb{R}$. So for each i, we have this decomposition of each interval $I_{1,i}$ as a union of disjoint intervals.

$$\begin{aligned} E =& \bigcup_{i=1}^{N} B_i \\ =& \bigcup_{i=1}^{N} \left(I_{1,i} \times I_{2,i} \times \cdots I_{n,i}\right) \\ =& \bigcup_{i=1}^{N} \left(\Big(\bigcup_{k_1=1}^{N_1} L_{1,i,k_1}\Big) \times \Big(\bigcup_{k_2=1}^{N_2} L_{2,i,k_2}\Big) \times \cdots \Big(\bigcup_{k_n=1}^{N_n} L_{n,i,k_n}\Big) \right) \\ =& \bigcup_{i=1}^{N} \bigcup_{k_1=1}^{N_1} \bigcup_{k_2=1}^{N_2} \cdots \bigcup_{k_n=1}^{N_n} \underbrace{\left(L_{1,i,k_1} \times L_{2,i,k_2} \times \cdots \times L_{n,i,k_n}\right)}_{B'_{i,k_1,k_2,\ldots,k_n}} \end{aligned}$$

$\{B'_{i,k_1,k_2,\ldots,k_n}\}_{k_1,\ldots,k_n,i}$ is a collection of disjoint boxes in $\mathbb{R}^n$.

Now we have the decomposition.

$$\begin{aligned}
E &= \bigcup_{i=1}^{N} B_i \\
&= \bigcup_{i=1}^{N} I_{1,i} \times I_{2,i} \times \cdots \times I_{n,i} \\
&= \bigcup_{i=1}^{N} \Big((\bigsqcup_{k_1=1}^{N_1} L_{1,i,k_1}) \times (\bigsqcup_{k_2=1}^{N_2} L_{2,i,k_2}) \times \cdots \times (\bigsqcup_{k_n=1}^{N_n} L_{n,i,k_n}) \Big) \\
&= \bigcup_{i=1}^{N} \bigcup_{k_1=1}^{N_1} \bigcup_{k_2=1}^{N_2} \cdots \bigcup_{k_n=1}^{N_n} (L_{1,i,k_1} \times L_{2,i,k_2} \times \cdots \times L_{n,i,k_n}).
\end{aligned}$$

Note here that

$$B'_{i,k_1,k_2,\cdots,k_n} = L_{1,i,k_1} \times L_{2,i,k_2} \times \cdots \times L_{n,i,k_n}$$

is a box and all these boxes are disjoint. So,

$$\left\{ B'_{i,k_1,k_2,\cdots,k_n} \right\}_{k_1,k_2,\cdots,k_n,i}$$

is a collection of disjoint boxes. Thus, we have expressed E as a collection of, as a finite union of disjoint boxes in $\mathbb{R}^n$.

So, it is a little bit tedious, but the geometric intuition is clear that we have to separate our coordinates, express the union of these intervals in each coordinate as a union of disjoint intervals and then breakage interval as a union of these disjoint intervals and again take the Cartesian coordinate.

(ii) To show: If $E = \bigcup_{i=1}^{N} B_i = \bigcup_{j=1}^{M} B_j'$

$\{B_i\}_{i=1}^{N}$ is a collection of disjoint boxes in $\mathbb{R}^n$

$\{B_j'\}_{j=1}^{M}$ " " "

then $\sum_{i=1}^{N} m(B_i) = \sum_{j=1}^{M} m(B_j')$

To show: $m(B_i) = \lim_{N \to \infty} \frac{1}{N^n} \# \left\{ B_i \cap \frac{\mathbb{Z}^n}{N} \right\}$

$\mathbb{Z}^n = \{ (k_1, k_2, \dots, k_n) : k_i \in \mathbb{Z} \}$

Now come to the second point. We have to show that if E is a disjoint union written in two ways, let say

$$E = \bigsqcup_{i=1}^{N} B_i = \bigsqcup_{j=1}^{M} B'_j,$$

where $\{B_i\}_{i=1}^{N}$ are collection of disjoint boxes in $\mathbb{R}^n$, and $\{B'_j\}_{j=1}^{M}$ are collection of disjoint boxes in $\mathbb{R}^n$. Then, we have to show that

$$\sum_{i=1}^{N} m(B_i) = \sum_{j=1}^{M} m(B'_j).$$

Now, one way to do this will be to get a collection of disjoint boxes which refine both these collection. Which means that each B_i's and B'_j can be written as a finite union of even smaller disjoint boxes and then we can use interchange of summations to two pieces. But I am not going to use that way. I am going to show that

$$m(B_i) = \lim_{N\to\infty} \frac{1}{N^n} \# \Big\{ B_i \cap \frac{\mathbb{Z}^n}{N} \Big\},$$

where

$$\mathbb{Z}^n = \{(k_1, k_2, \cdots, k_n) : k_i \in \mathbb{Z}\}.$$

for $n = 1$: Let I be an interval.

To show:

$$m(I) = \lim_{N\to\infty} \frac{1}{N} \# \Big\{ I \cap \frac{\mathbb{Z}}{N} \Big\}$$

$$\frac{\mathbb{Z}}{N} = \Big\{ \frac{k}{N} : k \in \mathbb{Z} \Big\}$$

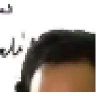
Suppose $I = [a, b]$

$$I \cap \frac{\mathbb{Z}}{N} = \Big\{ \frac{k}{N} \in I : k \in \mathbb{Z} \Big\} \to \# \Big\{ I \cap \frac{\mathbb{Z}}{N} \Big\}$$

$$\Rightarrow \quad a \leq \frac{k}{N} \leq b$$

$$\Rightarrow \quad Na \leq k \leq Nb.$$

how may integers satisfy this inequali



So let us see what it means for $n = 1$. Let I be an interval. we have to show that

$$m(I) = \lim_{N \to \infty} \frac{1}{N} \# \Big\{ I \cap \frac{\mathbb{Z}}{N} \Big\}.$$

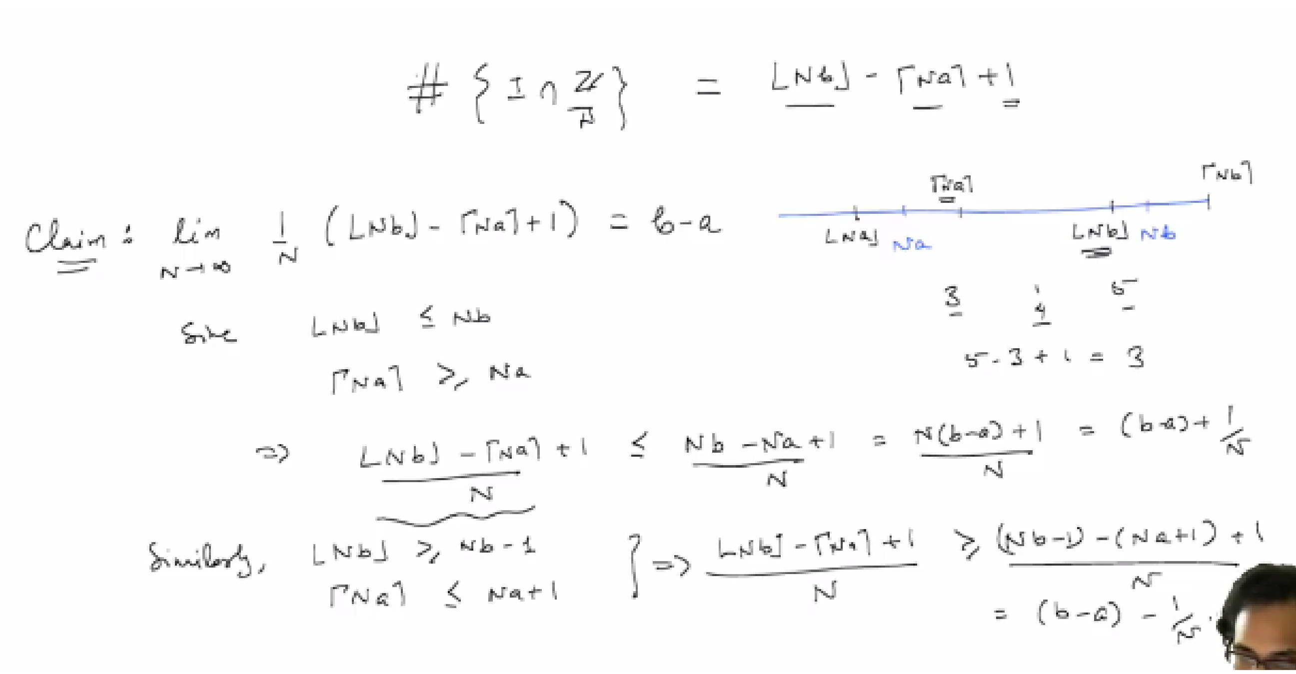
Note that

$$\frac{\mathbb{Z}}{N} = \Big\{ \frac{k}{N} : k \in \mathbb{Z} \Big\}.$$

Suppose $I = [a, b]$, a close interval. You can do this same proof or if one of the end points is open or if both are open, but let us see for the case when both are closed. Then,

$$I \cap \frac{\mathbb{Z}}{N} = \Big\{ \frac{k}{N} \in I : k \in \mathbb{Z} \Big\}.$$

Note that $a \leq \frac{k}{N} \leq b$. Which imply $Na \leq \frac{k}{N} \leq Nb$. So we have to count how many integers k satisfy this inequality.



So, this is the numbers of integers between the points $Na$and Nb. A moment start will give you the following formula:

$$\# \Big\{ I \cap \frac{\mathbb{Z}}{N} \Big\} = \llcorner Nb \lrcorner - \ulcorner Na \urcorner + 1.$$

Note that Na lies between $Na - 1$ and Na.

Then $\llcorner Na \lrcorner$ is this largest integer less than or equal to Na and $\ulcorner Na \urcorner$ is the smallest integer greater than or equal to Na. Similarly for Nb.

We have to count the number of integers between the two. Since $\llcorner Na \lrcorner$ and $\ulcorner Na \urcorner$ are integers, the number of integer points between these two must be $\llcorner Nb \lrcorner - \ulcorner Na \urcorner + 1$. For example if this is $\llcorner Na \lrcorner = 3$ and $\ulcorner Na \urcorner = 5$, then integer in between 3 and 5 are 3, 4, and 5. Thus, the number of integer in between 3 and 5 is $5 - 3 + 1$.

Now I have to take the limit as $N \to \infty$ and to show

$$\lim_{N\to\infty} \frac{1}{N}(\llcorner Nb \lrcorner - \ulcorner Na \urcorner + 1) = b - a.$$

Since $\llcorner Nb \lrcorner \leq Nb$ and $\ulcorner Na \urcorner \geq Na$, we have

$$\llcorner Nb \lrcorner - \ulcorner Na \urcorner + 1 \leq Nb - Na + 1.$$
$$\Rightarrow \frac{\llcorner Nb \lrcorner - \ulcorner Na \urcorner + 1}{N} \leq \frac{Nb - Na + 1}{N} = b - a + \frac{1}{N}.$$

Similarly, $\llcorner Nb \lrcorner \geq Nb - 1$ and $\ulcorner Na \urcorner \leq Na + 1$. Thus, we have

$$\frac{\llcorner Nb \lrcorner - \ulcorner Na \urcorner + 1}{N} \geq \frac{(Nb-1) - (Na-1) + 1}{N} = b - a - \frac{1}{N}.$$

$$(b-a) - \frac{1}{N} \leq \frac{\lfloor Nb \rfloor - \lceil Na \rceil + 1}{N} \leq (b-a) + \frac{1}{N}$$

$$\Rightarrow \lim_{N\to\infty} \frac{\lfloor Nb \rfloor - \lceil Na \rceil + 1}{N} = (b-a) = m(I)$$

Let $B = \bigcup_{i=1}^{m} I_i$ (disjoint union)

$$\lim_{N\to\infty} \frac{1}{N} \#\{B \cap \tfrac{\mathbb{Z}}{N}\} = \frac{1}{N}\sum_{i=1}^{m} \#\{I_i \cap \tfrac{\mathbb{Z}}{N}\}$$

$B = \bigcup_{j=1}^{k} I'_j$

$$= \sum_{i=1}^{m} \lim_{N\to\infty} \frac{1}{N} \#\{I_i \cap \tfrac{\mathbb{Z}}{N}\}$$

$$\sum_{j=1}^{k} m(I'_j) = \lim_{N\to\infty} \frac{1}{N}\#\{B \cap \tfrac{\mathbb{Z}}{N}\} = \underbrace{\sum_{i=1}^{m} m(I_i)} =: m(B)$$

Compare both the above equation, then we have

$$b - a - \frac{1}{N} \leq b - a - \frac{1}{N} \leq \frac{\llcorner Nb \lrcorner - \ulcorner Na \urcorner + 1}{N} \leq b - a + \frac{1}{N}.$$

So now take the limits on all three sides. This easily gives you that

$$\lim_{N\to\infty} \frac{1}{N}(\llcorner Nb \lrcorner - \ulcorner Na \urcorner + 1) = b - a = m(I).$$

Now if you have a disjoint union, so let say

$$B = \bigcup_{i=1}^{m} I_i \quad \text{(disjoint interval)},$$

then we have

$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{N} \#\Big\{B \cap \frac{\mathbb{Z}}{N}\Big\} &= \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{m} \#\Big\{I_i \cap \frac{\mathbb{Z}}{N}\Big\} \\
&= \lim_{N\to\infty} \sum_{i=1}^{m} \frac{1}{N} \#\Big\{I_i \cap \frac{\mathbb{Z}}{N}\Big\} \\
&= \sum_{i=1}^{m} \lim_{N\to\infty} \frac{1}{N} \#\Big\{I_i \cap \frac{\mathbb{Z}}{N}\Big\} \\
&= \sum_{i=1}^{m} m(I_i).
\end{aligned}$$

In the last line we are able to take the sum inside, because we are in finite sum. Therefore, if we have a disjoint union for the case $n = 1$, we have shown that $\sum_{i=1}^{m} m(I_i)$ can be expressed entirely in terms of the union B. Therefore if two unions are the same, meaning that if

$$B = \bigcup_{j=1}^{k} I'_j,$$

then also we have

$$\sum_{j=1}^{k} m(I'_k) = \lim_{N\to\infty} \frac{1}{N} \#\Big\{B \cap \frac{\mathbb{Z}}{N}\Big\} = \sum_{i=1}^{m} m(I_i).$$

So the sum does not depend on the decomposition of B into various disjoint intervals. So we call this common value by definition as $m(B)$.

So one can easily generalize this same argument in n dimensions where you work coordinate wise and you will get the same answer. So, I leave that as an exercise and we stop our lecture here. In the next lecture, we will see some properties of this elementary measure, which is defined using the sum of disjoint boxes.