Measure Theory
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Lecture - 49
L 1 Functions on R d: Proof of Lusin's Theorem, Space of L 1 Functions as a Metric Space

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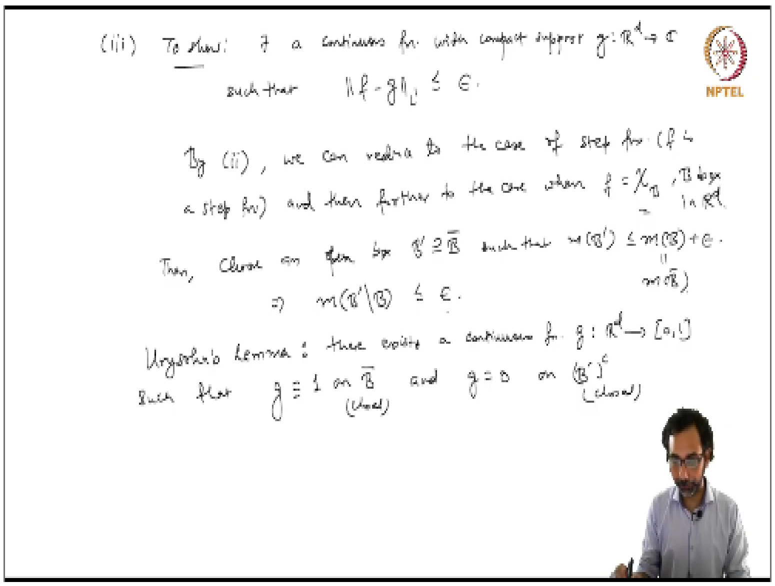

Now, for the proof of the third part in which we have to show that there exists a continuous function with compact support g complex valued such that the norm of f - g is less than or equal to epsilon. So, here we will use the second part. So, by the second part, we can reduce to the case of step functions meaning that f is a step function and then further to the case when f is the indicator function of a box b.

So, using the same kind of linearity trick that we used before using triangle inequality. We can reduce to this case when f is simply the indicator function of a box. So, then choose an open elementary set, open box actually, open box B prime containing B such that the measure of B prime is less than or equal to the measure of B plus epsilon which means that the measure of B prime - B is less than or equal to epsilon.

Here, I would also like it to contain the closure of B. So, our box may not be closed but we can arrange. So, that B prime contains not only the box but also its closure and then we will still have this inequality because the measure of B is the measure of B bar. So, we can assume without loss of generality that B is a closed box and now, we have chose chosen an open box B prime containing this closed box B.

And now, I am going to use Urysohn's Lemma which states that there exists a continuous function g from R d to the set 0 1. So, it takes values only in the interval 0 1 such that g is equal to 1 on the closed set B bar B closure and g = 0 on B prime complement. So, this is about separation of closed subs disjoint closed subsets of R d. This is a closed set and this is a closed set.

So, Urysohn's Lemma states that there exists a continuous function which is one on one of the closed sets and zero on the other closed set. So, we are going to use this to determine our g.

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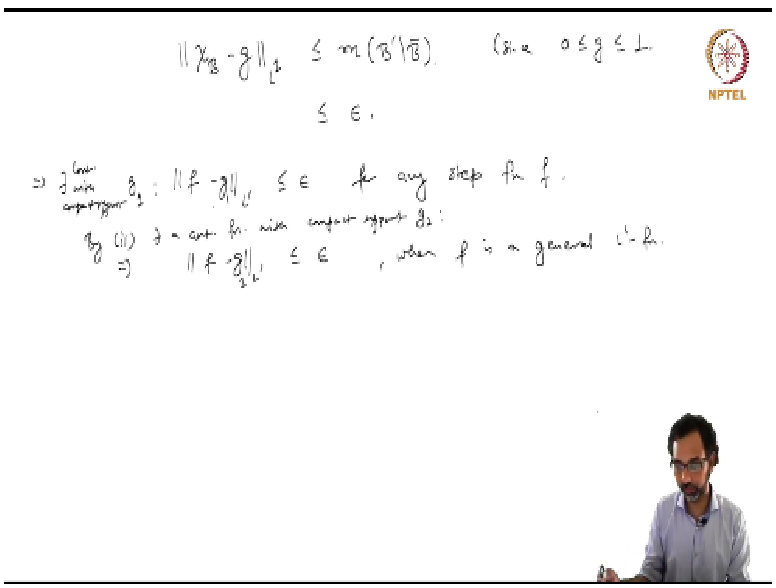

So, in fact this is the g we want, because the norm of chi B minus this g and the L 1 norm is bounded above by the measure of B prime - B bar because otherwise g is 1 on B. Okay. And then the only place where g is not equal to 0 is then outside B but then it is again the 0 outside B prime. So, since g is g takes the value between 0 and 1 it is less than or equal to 1 times the measure of the set on which g is strictly between 0 and 1.

So, this is what we get and this is less than or equal to epsilon. So, this implies that we have f - g is less than or equal to epsilon for any step function f and this implies again by 2 that f - g is less than or equal to. So, here this g is not the same as this g. So, let me write g 1 and g 2. So, here there exists a continuous function with compact support g prime g 1 such that this happens. And here there exists a continuous function with compact support g 2 such that f – g 2 has L 1 norm less than or equal to epsilon when f is a general step function either a general L 1 function.

So, we have reduced it to various cases and we have shown this; all the three parts of the preparatory lemma.

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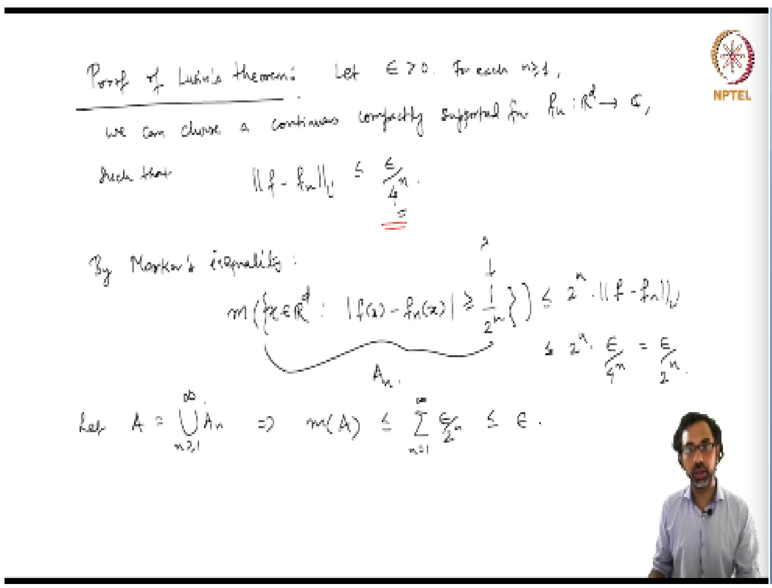

And now we come to the proof of Lusin's theorem. So, let us fix epsilon greater than 0 and by our preparatory lemma for each n, we can choose a continuous compactly supported function f n from R d to c such that the L 1 norm of f - f n is less than or equal to epsilon by 4 to the power n. So, you will see why we have chosen this epsilon by 4 to the power n but this is our function g which has chosen.

We can choose it to be continuous with compact support and our epsilon in this case is epsilon **(( )) (08:05)** by 4 to the power n. So, this g since this depends on n, we have renamed it as f n and of course, we would like our f n to converge to f. So, rather than using g's, we are using f n's and now by Markov's inequality, we have that the measure of points in R d such that f x - f n x less than or equal to sorry greater than or equal to 1 over 2 to the power n.

So, the measure of this set is less than or equal to. So, this is our lambda here. So, it is less than or equal to 1 over lambda times the L 1 norm of the function that you are choosing. So, this is nothing but 2 to the power n times the L 1 norm and because we have chosen this L 1 norm to be less than or equal to epsilon by 4 to the power n. So, then this is 2 to the power n times epsilon by 4 to the power n which is equal to epsilon by 2 to the power n.

So, now let me denote this set by A n and let A be a union of A n s; n greater than equal to 1. So, this means that the measure of A is less than or equal to the sum epsilon by 2 to the power n; n = 1 to infinity and this is less than or equal to epsilon.

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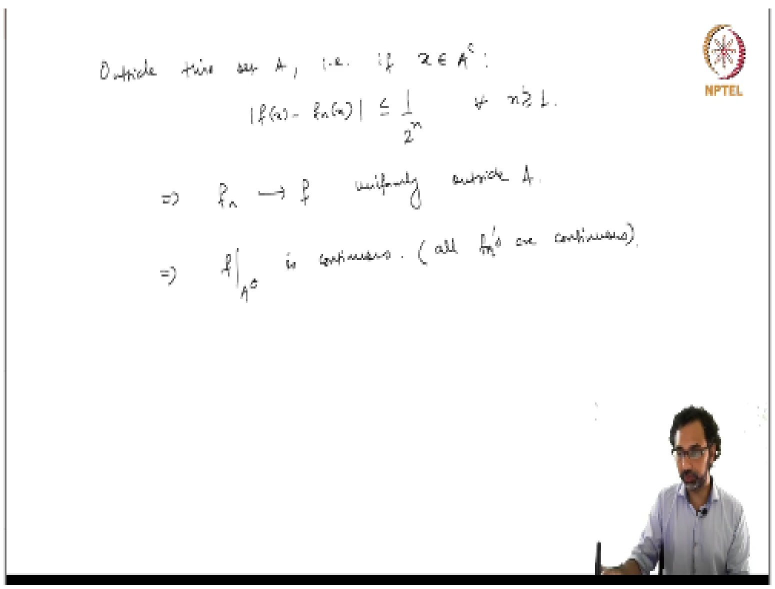

So, the measure of A is less than or equal to epsilon and let us see what happens outside of A. So, outside this set A. So, if x belongs to A complement; then we have f x - f n x is less than or equal to 1 over 2 to the power n for all n greater than or equal to 1. And this implies uniform convergence of this function sequence f n converges uniformly to f outside A. And so, this is our required set outside of which there is uniform convergence of f n's to this function f.

This also implies that f restricted to A complement is continuous because all these f n's are continuous. So, uniform convergence of continuous functions gives you a continuous function. So, this means that f when restricted to the complement of this set A is a continuous function which proves Luzin's theorem.

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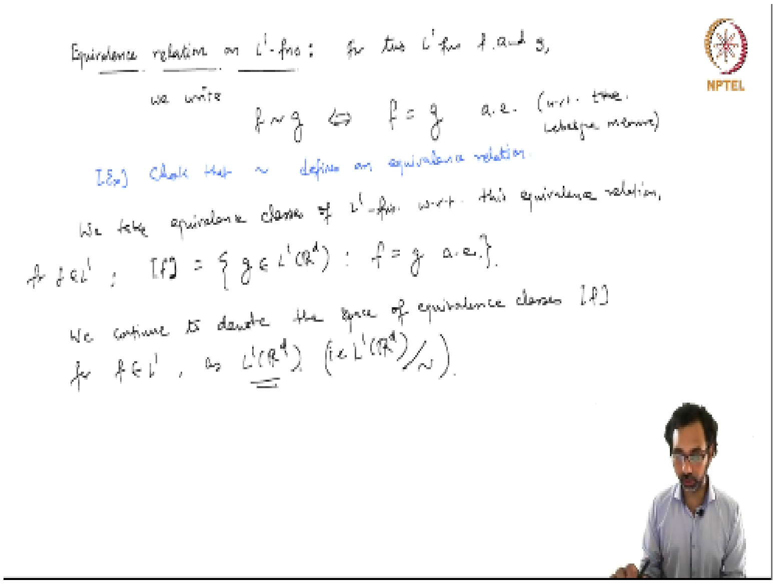

Now, we put an equivalence relation on L 1 functions as follows. So, we write for 2 L 1 functions f and g. We write f is equivalent to g if and only if f agrees with g almost everywhere. So, again with respect to the Lebesgue measure. So, we are on still on R d and with this equivalence. So, first of all one has to check that this is an equivalence relation. It is quite easy to check that this defines and equivalence relation.

So, it is almost immediate from the definition of the equal and the relation that it is an equivalence relation. So, now we take the equivalence classes of functions L 1 functions with respect to this equivalence relation. And so, we write the equivalence class of f. So, for f in L 1, the equivalence class of f is the space of all L 1 functions such that f = g almost everywhere. So, rather than using a new notation for this new space of equivalence classes we continue to denote the space of equivalence classes f for L 1 functions as L 1 of R d.

So, rather than using a new notation we still write it as L 1 R d which actually should have been written as L 1 R d from our previous notation modulo this equivalence relation. So, rather than writing it as modulo always we write, we are going to write it as simply L 1 of R d.

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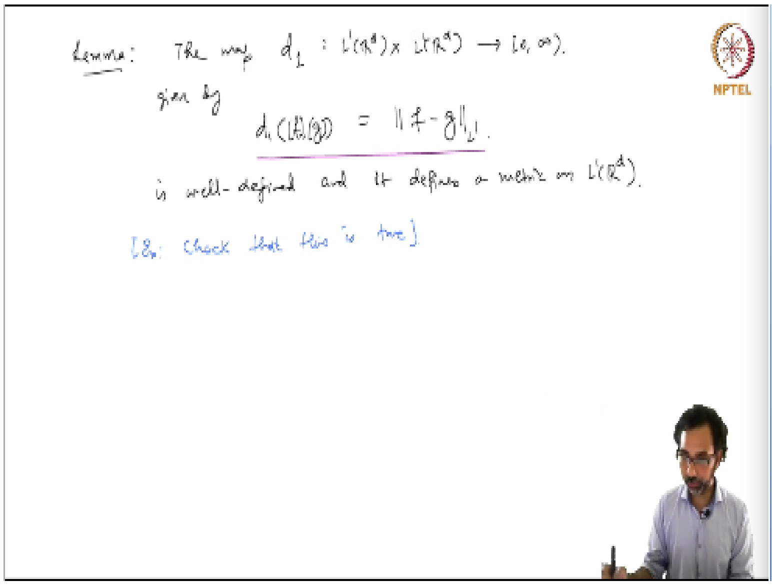

Now, with this new space, we have a following lemma that the map d 1 from L 1 R d cross L 1 R d to 0 infinity given by d 1 of f comma g. Actually, these are equivalence classes but it is not going to make any difference. So, this is the simply the difference of the L 1 norms of f and g. So, one has to check that even if you take f prime and g prime in the same equivalence class; you will get the same result but this is an obvious statement to make.

So, we have that first that this is a well-defined function is well defined and it defines a metric on L 1 R d. So, by the properties of the metric, we just have to check for positivity and triangle inequality and reflectivity. So, I leave it as an exercise again to check that this is true which is that this defines a metric on the L 1 functions after you have modulo out by the equivalence relation.

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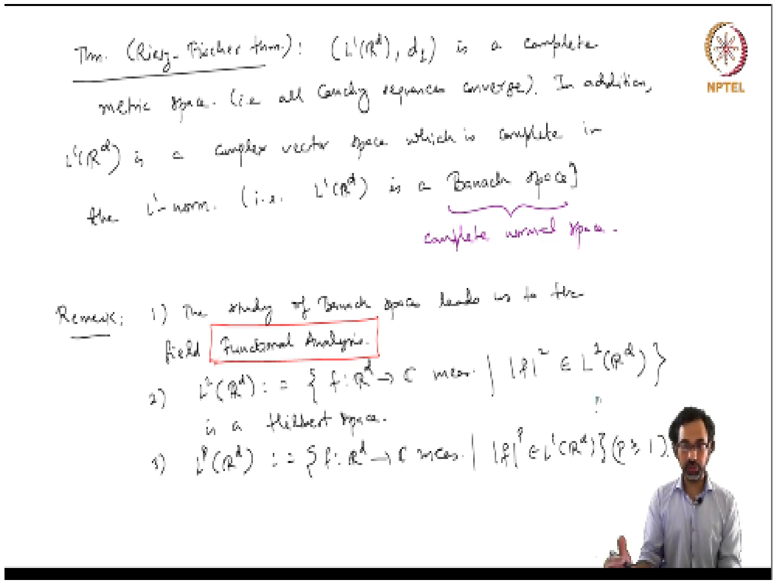

So, the importance of this metric comes from the next theorem which is that which is called the Riesz-Fischer theorem and it says that L 1 R d equipped with this metric d 1 is a complete metric space which means that all Cauchy sequences converge. So, we also know that additions of L 1 functions are in L 1 and scalar multiplication by complex numbers of L 1 functions is also in L 1. So, in addition to being just a complete matrix space, we can say that it is a complete.

It is a complex vector space. So, here L 1 R d is a complex vector space which is complete in the L 1 norm. So, this is a complete normed space which means that L 1 R d is a Banach space. So, remember that the Banach space is a complete normed space. So, L 1 R d with the L 1 norm is then a complete normed space and so, a Banach space. So, this study of Banach spaces is. So, let me just remark here that the study of Banach spaces leads us to the field of functional analysis.

And in particular, the space of L 2 functions on R d which is defined as the space of measurable functions, complex measurable functions, measurable such that modulus f square belongs to L 1 of R d. This is a so called Hilbert space, is a Hilbert space. So, I have not defined the norm here but I just want to mention that this L 2 space and a L 2 space is a; has a special importance in the space of these kinds of Banach spaces of L P functions.

So, we can also define L P of R d which is the same class of complex measurable functions such that the Pth power. So, here P is a real number greater than 1 and this is a also Banach space.

So, all of these L P spaces are Banach spaces but in particular, this space L 2 is a the only one which is a Hilbert space. So, all these things are topics that are taken up in the field of functional analysis.

So, we do not have time to go into all these details but the Riesz-Fischer theorem is a very fundamental first result in this area which says that L 1 R d is a Banach space.