Measure Theory

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Lecture - 48

L 1 Functions on R D: Statement of Lusin's Theorem (Littlewood's Second Principle), Density of Simple Functions, Step Functions, and Continuous Compactly Supported Functions in L 1

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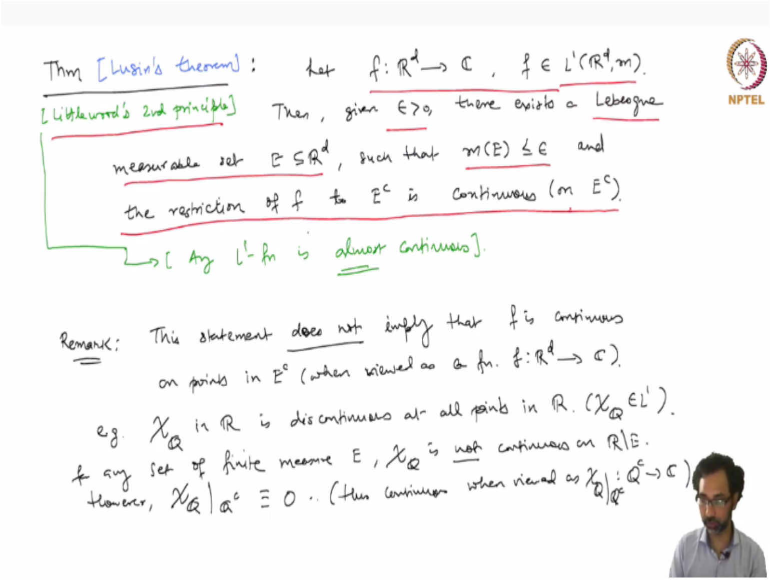

Now we come to Lusin's theorem, as I mentioned before, this is Littlewood's second principle. The precise version of Littlewood's second principle and the statement is that if f is an L 1 complex valued L 1 function on R d. When given any epsilon greater than 0, there exists a Lebesgue measurable set E such that the measure of E is less than or equal to epsilon, and the restriction of f to E complement is continuous on E complement.

So, this is what is meant by saying that any L 1 function is almost continuous. So, this is Littlewood's second principle which says that any L 1 function is almost continuous, and by almost, we mean this precise statement that given any epsilon, you can find a set E of measure less than or equal to epsilon such that the restriction of F on the complement of that set is continuous on that set. So, one caution cautionary remark.

So, this statement of Lusin's theorem does not imply that f is continuous on points in E complement when viewed as a function from the whole of R d to C. So, it is not that f is a f is continuous at points of E complement when it is viewed as a function on from the whole of R d to C. But rather it is only the restriction of f to E complement, which is continuous on the restricted set E complement.

So, what is the difference? So, for example, the indicator function of the rational numbers in R is discontinuous at all points in R. And it is an L 1 function because the measure of the rationals is 0. So, chi Q is an L 1 function. On the other hand, for any set of finite measure E chi Q is not continuous on R – E, because it is not continuous anywhere in R when it is viewed as a function from R.

However the restriction of this indicator function when restricted to the complement of the rationals. This is identically 0, and therefore continuous when viewed from the complement of the rational sets. So, thus continuous when viewed as a map from, so this restricted map is from the complement of Q to C. So, this shows that you can have, you cannot have does not imply this statement of Lusin's theorem does not imply that f is continuous on points in E complement when viewed from the entire domain R d with values in complex **(()) (04:50)**.

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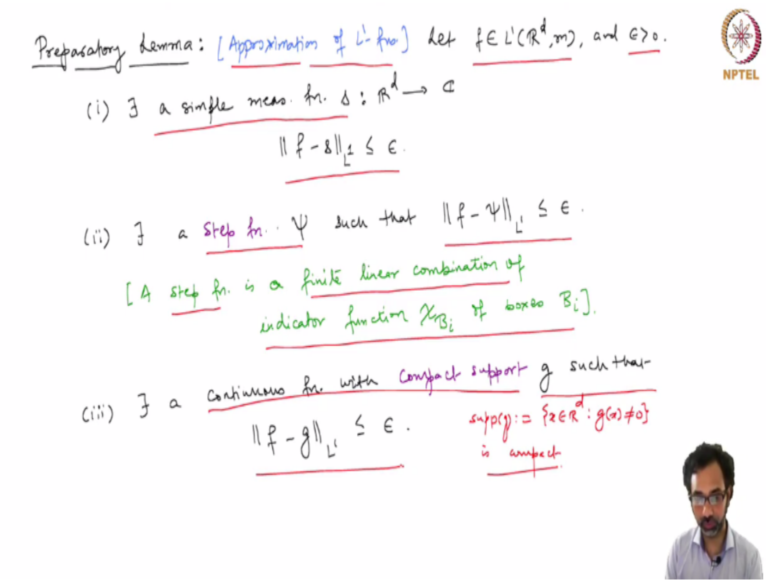

To prove Lusin's theorem, we need a preparatory lemma about approximation of L 1 functions. So, this is our lemma. And it is interesting in its own right. But here we will use it to prove Lusin's theorem. So, it has three parts so first we take an L 1 function f and fix an epsilon greater than 0, then the first part says that there exists a simple measurable function s such that the L 1 norm of f – s is less than or equal to epsilon.

Then the second part is that there exists a step function psi such that norm of f – psi L 1 norm is less than or equal to epsilon. Here, a step function is a finite linear combination of indicator functions of boxes B i. So, the difference between a simple measurable function and a step function is that simple measurable functions are defined with finite linear combinations of indicator functions of any measurable sets of finite or infinite measure.

But here for a step function, we only allow finite linear combination of indicator functions of boxes. So, it is very close to what we called piecewise constant functions, but here there is no partition of a single box and this is a more general concept than piecewise constant functions. So, these are called step functions. And the third part is that there is a continuous function with compact support such that the norm of f – g is less than or equal to epsilon.

So, remember that the compact support condition means that support of g which is defined as the set of points such that g x is not equal to 0. And you take the closure of this set. So, this is called the support of this function g and this is compact. So, when this is compact, we say that g is of compact support or g is a compactly supported function. So, in the third part, we have that there is a continuous function with compact support g such that the L 1 norm is less than or equal to epsilon.

So, all these three are approximations of L 1 functions in the L 1 norm with different kinds of different classes of functions. So, let us look at the proofs for these statements.

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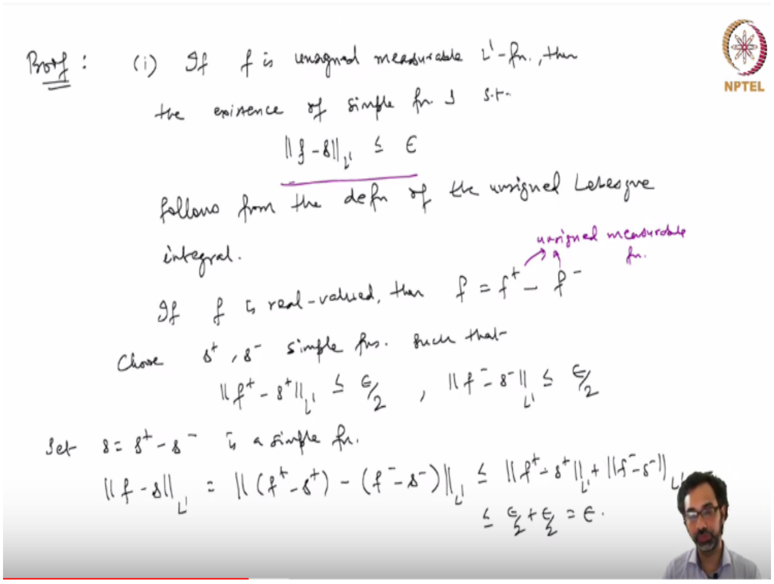

So, for the first part, we note that if f is unsigned. So, in the case of unsigned measurable function, measurable L 1 function then the existence of s, the existence of a simple function s such that the L 1 norm less than or equal to epsilon is follows from the definition of the Lebesgue unsigned **(()) (08:30)** integral. Because the unsigned Lebesgue integral was defined as a supremum of all simple functions which are bounded above point wise by f.

And you take the simple Lebesgue integral of those simple functions and you get when you take the supremum you get the Lebesgue integral of f. So, this is almost by definition of the Lebesgue integral for unsigned measurable functions. Now if f is real valued, then we can write f as f + – f – in the positive and negative parts, and then choose. Because these are now unsigned measurable functions.

Both of these are unsigned measurable functions and then we can use the part we have already shown for unsigned measurable functions. So, now choose s + and s – simple functions. So, whenever I say simple functions and meaning simple measurable simple functions such that norm of f + – f s + L 1 norm is less than or equal to say epsilon by 2. And similarly, for s – and f –, the L 1 norm is less than or equal to epsilon by 2.

And then if we set s as s + – s – then this is a simple function. And you have f – s the L 1 norm is equal to f + – s + – f – – s –. So, if we take the L 1 norm by triangle inequality, we will have that this is less than or equal to f + – s + + f – – s – L 1 norms. And these are less than or equal to epsilon by 2 each. So, this is less than or equal to epsilon. So, once we have shown this for unsigned measure functions, the real valued case follows almost immediately.

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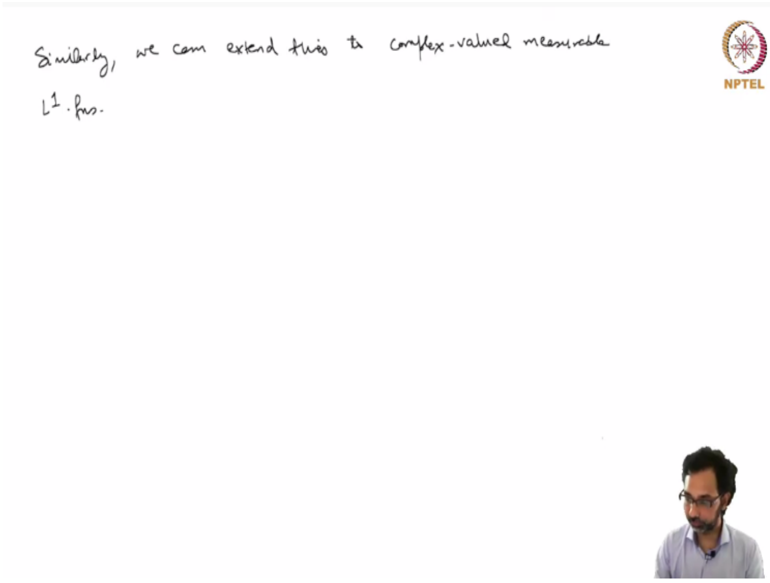

And similarly we can. Similarly, we can extend this to complex measurable functions, measurable L 1 functions. So, notice that here.

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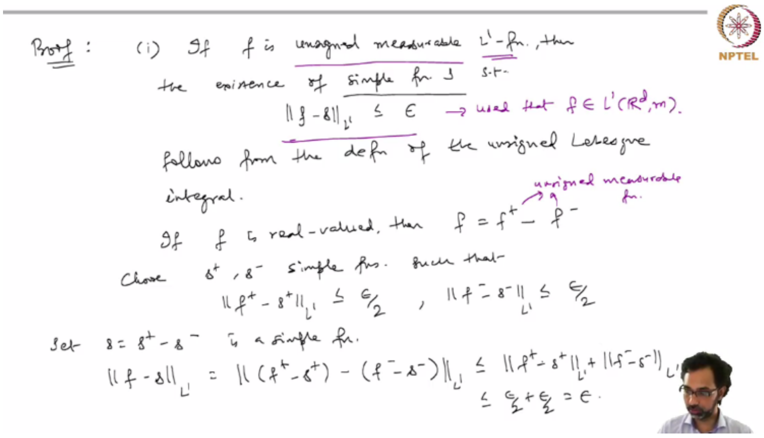

We need even for the existence of the simple function for the unsigned case for the unsigned measurable case, we need an L 1 function, because we need the Lebesgue integral to be finite in order to write this inequality of f – s L 1 norm of f – s less than or equal to epsilon. So, here we have used that f is an L 1 function.

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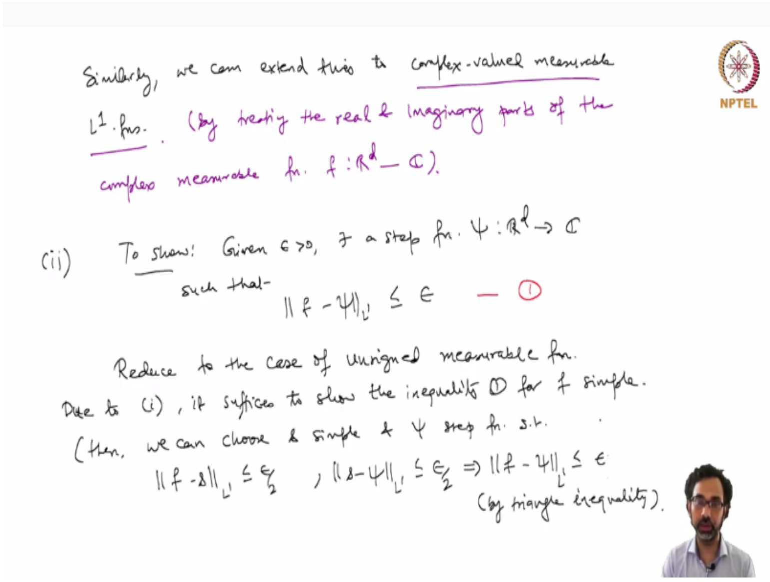

So, we have shown that we can extend this to complex valued L 1 functions as well, just by separately treating the real val real part and imaginary part of the complex measurable function by treating the real and imaginary parts of the complex function, complex measurable function, f. So, this proves the first part. For the second part, we have to find a step function.

So, we have to show that given epsilon, there exist a step function psi from R d to C such that norm of f – psi L 1 norm is less than or equal to epsilon. So, again we can reduce to the case of unsigned measurable functions. And because of the first part due to one, it suffices to show the inequality. So, let me write it as one. So, the inequality 1 for that for f simple, if you have a simple function.

And if you show that there exists an step function which is close to a simple function, then we can deduce it for any unsigned measurable function because of the first part. Because then we can choose s simple and psi step function such that the norm of f – s is less than or equal to epsilon by 2, and norm of f s – psi is less than or equal to epsilon by 2 which implies that the norm of f – psi is less than or equal to epsilon by the triangle inequality. So, this triangle inequality, when I say triangle inequality, I do not mean.

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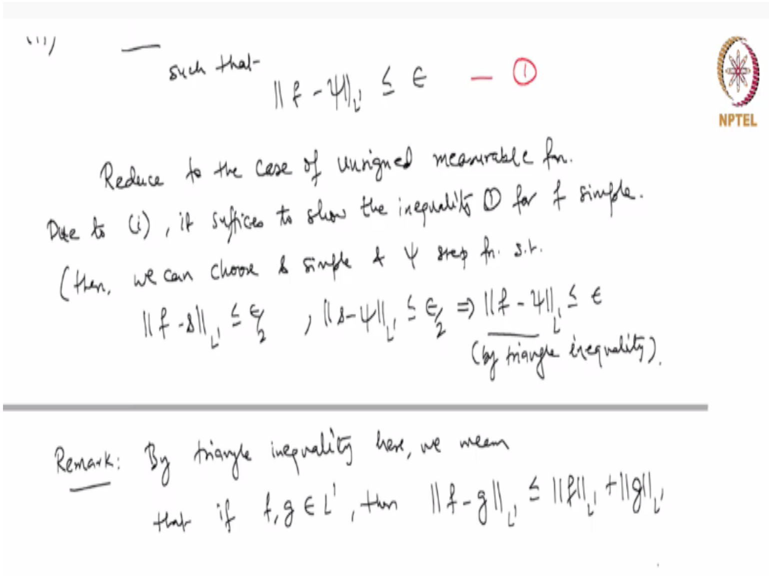

So let me write it here as a remark by triangle inequality here. We mean that if f and g are L 1 functions, then the norm of f + g f – g L 1 is less than or equal to norm of L 1 norm of f + L 1 norm of g. So, this is what we have used here to deduce that f – psi L 1 norm is less than or equal to epsilon. So, it suffices to show the case when we have reduced it to the case when f is simple.

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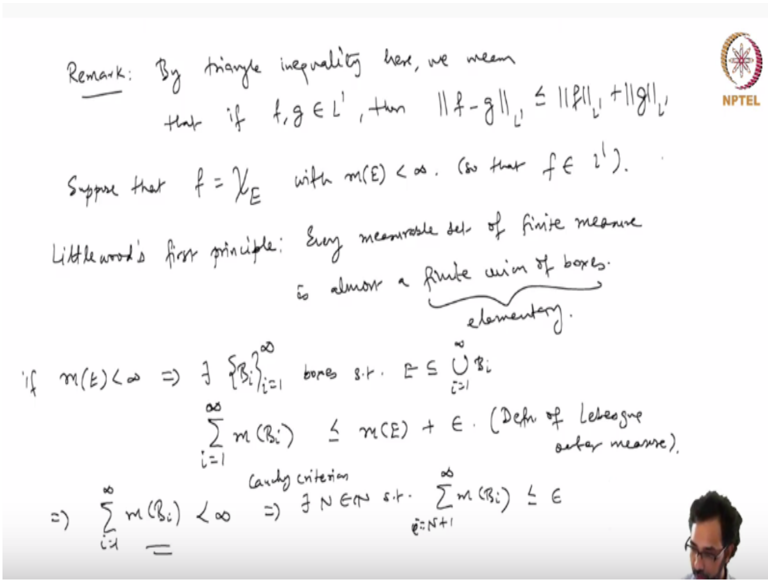

And now we further reduce it into the case when the simple function s is a single indicator function of a measurable set of finite measures. So, suppose that f is chi E with the measure being finite so that f is in L 1. So, this is a special case of a simple function which is just an indicator function of a set with measure with a finite measure. So, here again we are going to use Littlewood's first principle which is that every measurable set of finite measure is almost a finite union of boxes, or almost elementary.

This is elementary. So, how do we show this? So, let us give a short proof. So, if measure of E is finite. This implies that there exists a collection of boxes B i i = 1 to infinity, countably many boxes such that the sum. So, first that E is covered by these boxes, the union of these boxes. And then the sum i = 1 to infinity of the measures of these boxes B i is less than or equal to the measure of E + epsilon.

So, this is from the definition of the Lebesgue outer measure. And this also implies that this sum is finite, which means that by the Cauchy criterion of **(()) (19:08)** series. So, this is by the Cauchy criterion, there exists a number N such that the sum from N + 1 i = N + 1 to infinity m B i is less than or equal to epsilon. This is just from the convergence of this series, infinite sum m B i.

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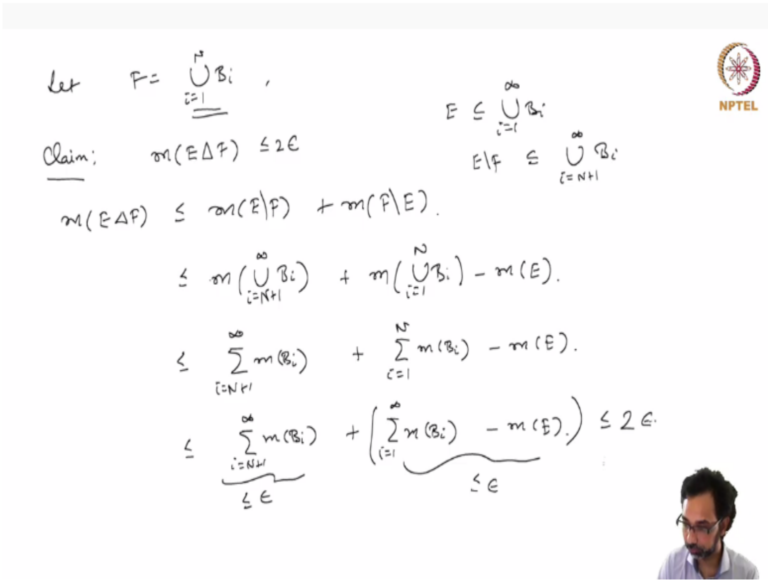

On the other hand, we can let F be the union from 1 to N of these B i's. And then I claim that the measure of E symmetric difference F is less than or equal to epsilon or 2 epsilon. Let us see what it is, I think it is 2 epsilon. So, how do we compute this? So, measure of E so this is the proof of the claim. This is less than or equal to the measure of E – F + the measure of F – E. And note that E was covered by this box **(()) (20:37)** B i.

And so when you leave out finitely many boxes. So, F was this finite union from 1 to N, then this is a subset of the union i = N + 1 to infinity of B i. So, therefore, we can write this as the measure of union i = 1 N + 1 to infinity B i + the measure of F – E which is i = 1 to N B i –

measure of E because both these are finite. So, we can simply and measurable. So, we can simply write it as a difference.

So, now this is less than or equal to the sum N +1 to infinity of m B i. And the next term is simply less than or equal to i = 1 to N m B i – m E. And this is again less than or equal to i = N + 1. So, the first term remains the same, but I let the second term go to infinity in the sum minus m E. Now note that this part is less than or equal to epsilon and this part is less than or equal to epsilon by the choice of B i. So, this is less than or equal to 2 epsilon.

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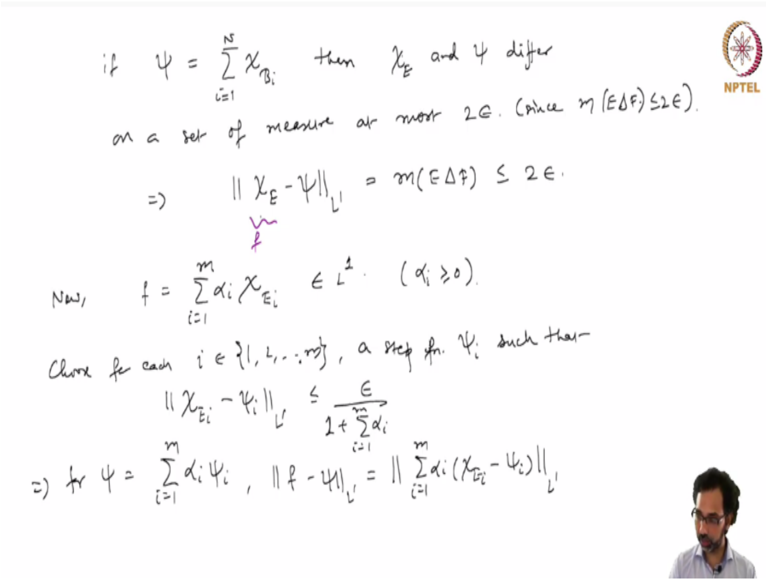

Now, if we let psi to be the sum i = 1 to N chi of B i, then chi E and psi differ on a set of measure at most 2 epsilon, since this set where they differ is precisely E delta F, and which is which has measure less than or equal to 2 epsilon. So, this means that the L 1 norm of chi E – psi which is by definition. This is now a simple function, this is equal to this measure of the symmetric difference of E and F. And this is less than or equal to 2 epsilon.

So, here, this is our F. And this is our psi. And we have proved this result of approximation by step functions when f is a single indicator function of a measurable set with finite measure. Now, we consider a more general L 1 simple function of the form 1 to m say alpha i chi of E i. So, this is not L 1 function. And now, choose for each i in 1, 2, up to m a step function psi i such that the norm of chi E i – psi i L 1 norm is less than or equal to epsilon over 1 + summation i = 1 to m of alpha i.

So, here note that alpha i’s are all positive real numbers and **(()) (25:04)** disjoint measurable sets, because f is a simple function. So, if we choose our psi i for each i such that this L 1 norm is less than or equal to epsilon by 1 + the sum of all these alpha i's. Then we have that for psi equals the sum alpha i psi i i = 1 to m the norm of f – psi is equal to the sum alpha i i = 1 to m c chi i chi E i – psi i. So, this is by construction. And now we can use triangle inequality.

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Using triangle inequality:

$$\|f-\psi\|_{L^1} \le \sum_{i=1}^{m} \alpha_i \underbrace{\|\chi_{E_i}-\psi_i\|_{L^1}}_{\le \epsilon / \left(1+\sum_{i=1}^{m}\alpha_i\right)}$$

$$\le \sum_{i=1}^{m} \alpha_i \frac{\epsilon}{1+\sum_{i=1}^{m}\alpha_i} = \underbrace{\left(\frac{\sum_{i=1}^{m}\alpha_i}{1+\sum_{i=1}^{m}\alpha_i}\right)}_{\le 1}\epsilon \le \epsilon$$

Similarly, extend this to real-valued & then complex-valued L^1 fns.

So, using triangle inequality, when the norm of f – psi is less than or equal to the sum i = 1 to m alpha I and then you have the norm of chi E i – psi i L 1 norm. And now these are less than or equal to epsilon by 1 + summation alpha i i = 1 to m. So, this is less than or equal to i = 1 to m alpha i epsilon over 1 + summation i = 1 to m alpha i. And now note that this is nothing but summation i = 1 to m alpha i over 1 plus something so 1 plus the same thing.

And then epsilon and this is less than or equal to epsilon because this whole thing is less than or equal to 1. So, from first from unsigned simple function with a single which is a single indicator function of a measurable set, we find a step function which is close to it. And then we can deduce the general case, simply by using triangle inequality. And then similarly, we can extend this to first real valued and then complex valued L1 functions. So, this shows, part two.