Measure Theory
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Lecture – 47
L 1 Functions on R d: Egorov's Theorem Revisited (Littlewood's Third Principle)

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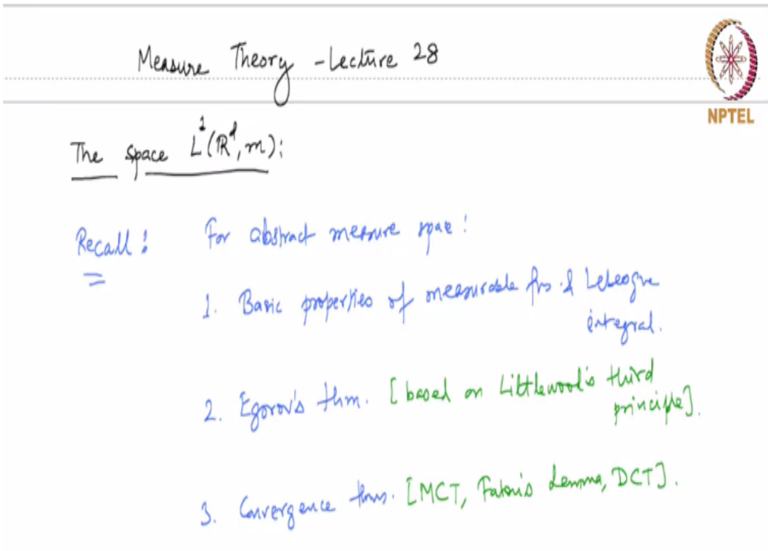

In the last lectures we have seen many different properties for the Lebesgue integral on abstract measure spaces. In this lecture we will go back to our initial space, which was the Euclidean space, R d with the, with our favorite Lebesgue measure, and we will see that there are some more interesting properties that hold in this special case. But let me recall first, what we have seen so far for abstract measure spaces.

So, for abstract measure space, we have seen, of course, we have seen the basic properties of the Lebesgue integral, basic properties of measurable functions and Lebesgue integral. We have also seen the abstract Egorov's theorem in the abstract set setting which was based on the Littlewood's third principle which was that a sequence of measurable functions converges uniformly outside set of negligible measure.

And we have also seen various convergence theorems. So, here we have seen Monotone Convergence Theorem Fatou's lemma and dominated convergence theorem, which are the three main pillars of Lebesgue integration theory. So, these all hold for the abstract measure space setting.

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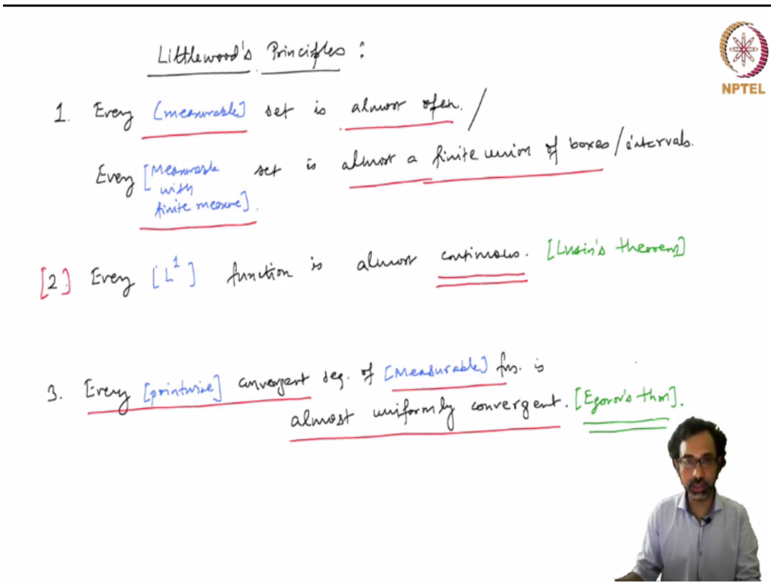

Now let us come back to Littlewood's three principles that we have seen before. So, the first one was that every measurable set is almost open. This was the definition of our notion of Lebesgue measurability for sets in R d. And we also had that every measurable set with finite measure is almost a finite union of boxes. So, I left it as an exercise, which is not very difficult to show that if you have a set of finite measure, then you can approximate it with a finite union of boxes.

So, this was the Littlewood's first principle and we have already seen this part. We have also seen Littlewood's third principle which says that every point wise convergent sequence of measurable functions is almost uniformly convergent. So, this was of course Egorov's theorem, which gave a for any sequence of measurable functions converging point wise to another function on a set of finite measure.

Then we can extract a set of negligible measure outside of which the convergence was uniform. So, this was the Littlewood's third principle. What we have not seen yet is Littlewood's second principle. So, this we have not seen the second principle which says that every L 1 function or absolutely integrable function is almost continuous. Now, note that we cannot state this second principle for arbitrary measure spaces because we need notion of continuity, which only holds for topological spaces.

And so this second principle we will state for the Euclidean space, R d. One can also state it for more general topological spaces like locally compact Egorov's spaces. But we will not state it here and we will only state it for R d. So, this is what is called Lusin's theorem and we will see this in today's lecture. But first, I will like to go back to Egorov's theorem and state a slightly more strengthened version of Egorov's theorem when we are dealing with the Euclidean space with the Lebesgue measure. So, for this let me make a couple of definitions.

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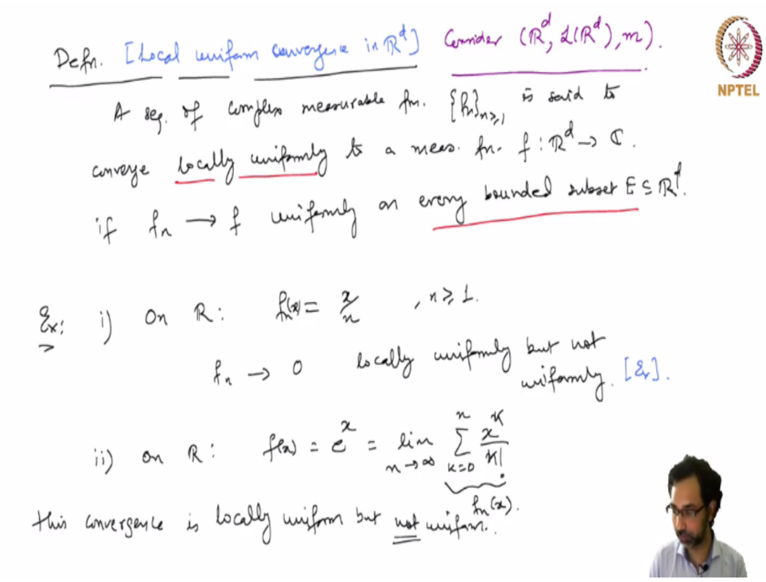

So, let me define the notion of local uniform convergence of functions, local uniform convergence in R d. Okay, so we are back to the case of the Euclidean space with the Lebesgue measure. Okay, so here, we consider the measure space, R d with the Sigma algebra of Lebesgue measurable sets and the Lebesgue measure. So, let us define what is local uniform convergence for R d. So, a sequence of measurable functions.

When complex measurable functions f n is said to converge locally uniformly to a measureable function f on R d again with values in the complex numbers if f n converges to f uniformly on every bounded subset E of R d. So, rather than asking for uniform convergence on the entirety of R d, we only ask it for having uniform convergence on every bounded subset of R d.

So, let us see an example of a sequence of functions converging local uniformly but not uniformly. So, first example is if you take f n so let me take on R. If you take f n to be the function x over n. Okay, and greater than equal to 1, then f n converges to the function zero locally uniformly but not uniformly. So, we see that this is a weaker notion of convergence.

And I leave it to you as an exercise to check that this convergence is not uniform, but it is locally uniform. Similarly if you take again on R, if you take the function f x to be the exponential function, and write it as the limit of the partial sums of its Taylor series K equal to 1 to 0 to n and you take x to the power n over x to the power K sorry over K factorial.

Then if you define these as your f n x, then this convergence is locally uniform but not uniform. So, this convergence is locally uniform, but not uniform. So, there are many interesting cases of functions sequences of functions converging local uniformly but not uniformly. So, let me state, a version of Egorov's theorem, which uses the notion of local uniform convergence in R d, and which slightly strengthens our abstract Egorov's theorem.

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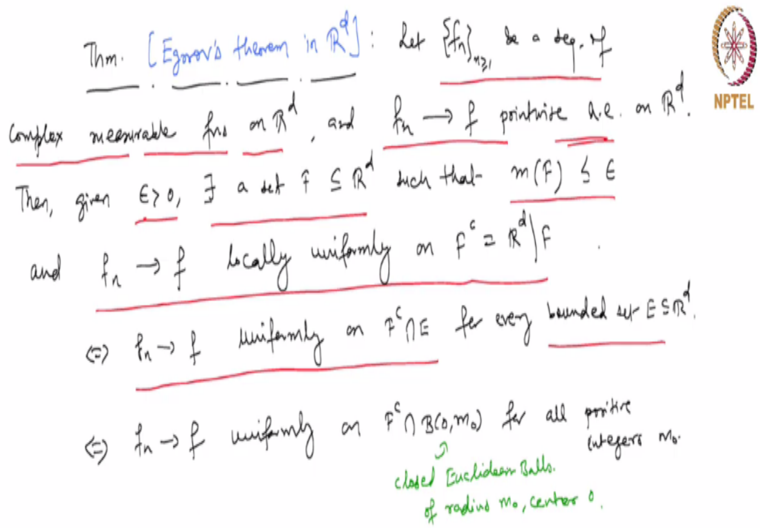

So, let us look at the statement of Egorov's theorem for R d. So, it states that if f n is a sequence of complex measurable functions on R d and it converges point wise to some function f point wise almost everywhere in R d. So, here we allow convergence point wise, almost everywhere. Then given epsilon greater than 0, there exists a set F such that the measure of the set is less than or equal to epsilon.

And f n converges to f locally uniformly on the complement of F, which is the same as saying that f n converges uniformly on f n F complement intersection some bounded set E for every bounded set E in R d. So, let me write another equivalent statement which is that f n converges to f uniformly on F complement intersection B 0 m not for all positive integers, m not.

So, we can check that this last statement is actually equivalent to the middle statement, which was for every bounded subset of E and but this is for closed Euclidean balls so this is the closed Euclidean balls of radius m not and center 0. So, we will prove this last statement to establish Egorov's theorem.

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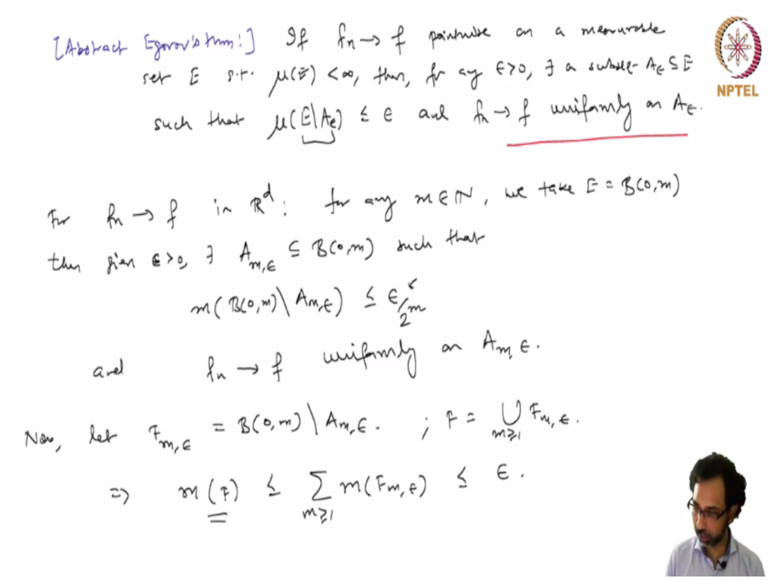

So, let us look at the proof, but first let me recall what was the abstract version of Egorov's theorem. So, in this case we had that if f n converges to f point wise. So, we did not assume point wise almost everywhere but we only assumed point wise convergence on a measurable set E such that it has finite measure. Then for any epsilon greater than 0, there exists a set subset A epsilon of E, such that the measure of E minus A epsilon is less than or equal to epsilon and f n converges to f uniformly on this set A epsilon.

So, outside this set A epsilon which has negligible measure inside E. We have uniform convergence on A epsilon itself. So, we are going to use this abstract version. So, for f n converging to f in R d for any positive integer m, we take E to be B 0 m in this above abstract version of Egorov's theorem. Then given epsilon greater than 0, there exists a set A m epsilon a subset of B 0 m such that first that the measure of B 0 m minus A m epsilon is less than or equal to epsilon.

And secondly that f n converges to f uniformly on A m epsilon. So, we can apply for any m this abstract Egorov's theorem and we deduce this statement. So, here I would like to take, rather than epsilon I would like to take epsilon over 2 to the power m, so that I can use in summation trick, as we have done before. So, now let us denote by F m epsilon, this set B 0 m minus A m epsilon.

And we take F to be the union of all these F m epsilons. Then of course the measure of F is less than or equal to the sum of all the F m epsilons and greater than or equal to 1. And so; this is less than or equal to epsilon, because each one is less than or equal to epsilon over 2 to the power m. So, we have that this measure of this set F is less than or equal to epsilon.

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Claim: For any $m_0 \in \mathbb{N}$, $f_n \to f$ uniformly on $F^c \cap B(0,m_0)$.

$$B(0,m_0)\cap F^c = B(0,m_0)\cap\Big(\bigcup_{m\geq 1} F_{m,\epsilon}\Big)^c$$
$$= B(0,m_0)\cap\Big(\bigcap_{m\geq 1} F^c_{m,\epsilon}\Big).$$
$$= B(0,m_0)\cap\Big(\bigcap_{m\geq 1} A_{m,\epsilon}\Big)$$

And now I claim that for any m not f n converges to f uniformly on F complement intersection B 0 m not. So, let us see why this is true. So, we have that B 0 m not intersection f complement is the same as B 0 m not intersection with so the complement of F. So, we had F itself as the union of these F m epsilons. And so you have to take the complement. So, this is B 0 m not intersection with the intersection now of m greater than or equal to 1, F the complement of F m epsilon.

But the complement of F m epsilon the intersection is just the same. This intersection is the same as the intersection of all A m epsilon.

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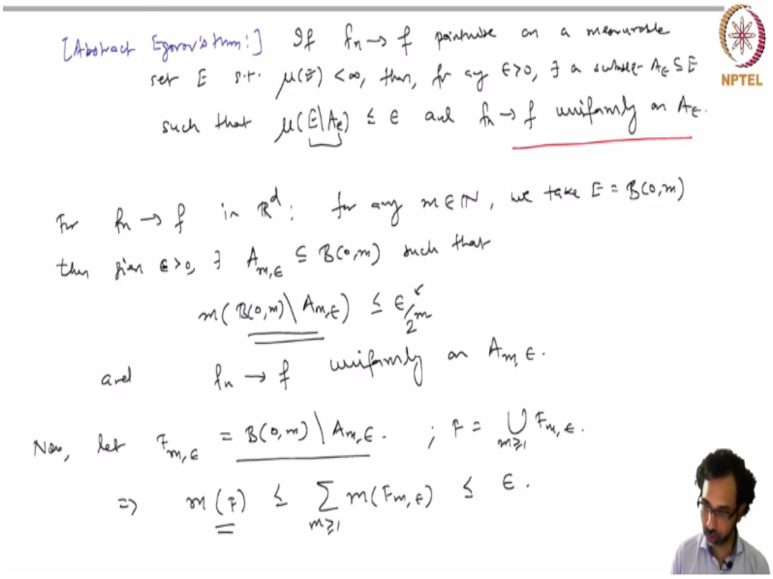

So remember that A m epsilon was the set we chose inside each B 0 m, the closed unit ball B 0 m and our F epsilon but was precisely the set B 0 m minus A m epsilon.

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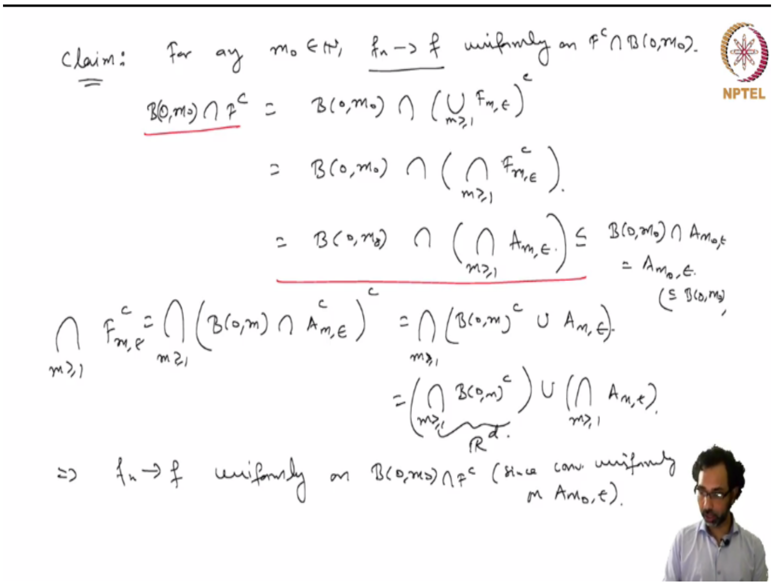

So, the complement of F m epsilon is. So, let me write it down here, F m epsilon complement is equal to B 0 m intersection A m epsilon complement and then you have a complement. So, then you get B 0 m complement union A m epsilon. So, if you take the intersection of all these. So, you should take the intersection over all m. Here also intersection over all m. And here also intersection over all m, then the first part intersection m greater than or equal to 1, B 0 m complement.

This is nothing but R d. So, you do not get anything extra here, and you will only get the intersection of A m epsilon. So, this is why we can write this last statement that B 0 m 0 intersection F complement is the same as B 0 m 0 intersection with the intersection of all A m epsilons. But now this set is a subset of B 0 m 0 intersection A m 0 epsilon in particular because we have an intersection over all m.

And this is nothing but A m 0 epsilon because this is a subset, A m 0 epsilon is a subset of B 0 m 0 by construction. So, on A m 0 epsilon, we have by construction that f n converges to f uniformly. And so this implies that f n converges to f uniformly on B 0 m 0 intersection F complement, since it converges uniformly on A m 0 epsilon. So, we see that we have local uniform convergence in R d when you have a sequence of functions converging point wise almost everywhere.

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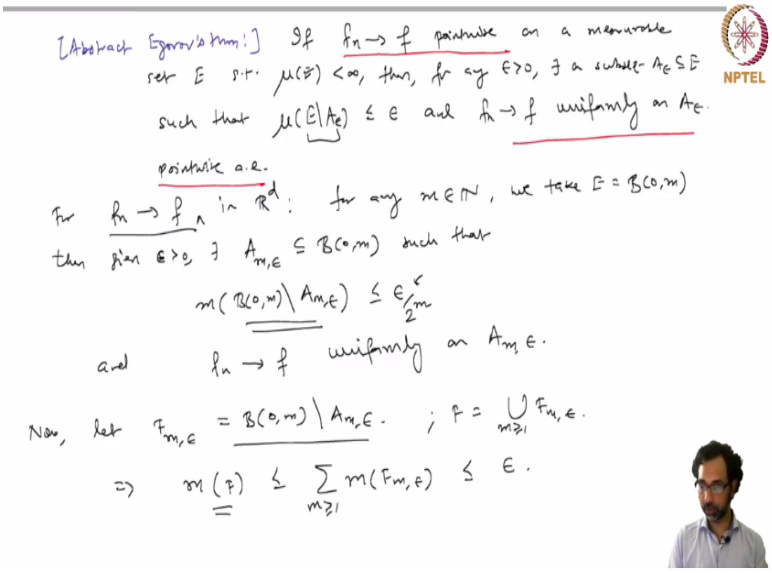

So, here I missed one thing, which is that our function converges point wise everywhere point wise almost everywhere. Here we have point wise almost everywhere convergence but here we need point wise everywhere convergence. But this is not an issue because we can modify this point wise convergence states that it converges everywhere when the modification is on a set of measure 0. And then we can add this set to our set F at the end, so that we still have a set of measure less than or equal to epsilon.

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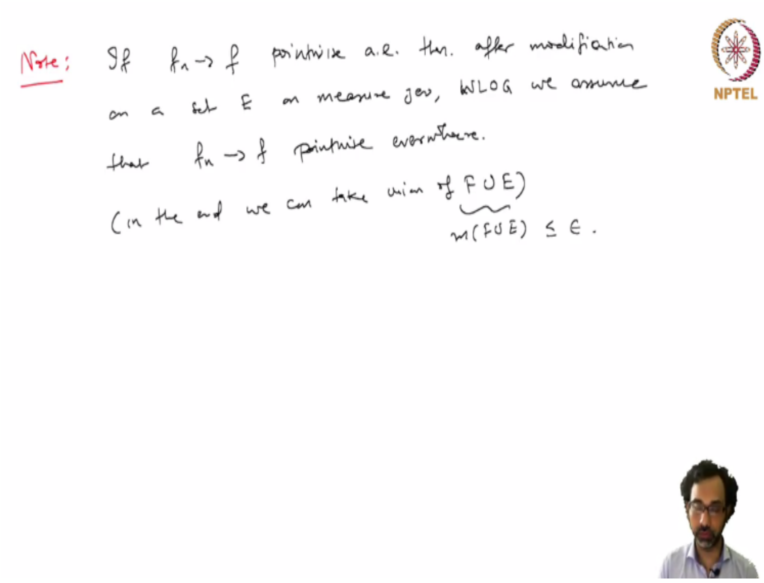

So, note that if f n converges to f point wise almost everywhere, then after modification on a set let us call it E of measure 0. Without loss of generality, we can assume that f n converges to f point wise everywhere rather than almost everywhere. And in the end, we can take the union of F union this set of measure 0. And this set, still has measure less than or equal to epsilon. So, a modification on a set of measure 0 does not affect our statement of Egorov's theorem.