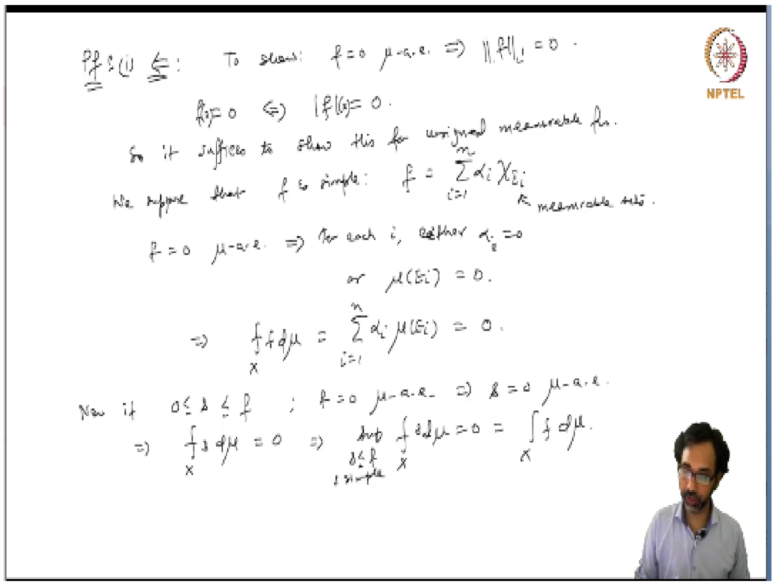
Measure Theory
Prof. Indrava Roy
Department of Mathematics
Institute of Mathematical Science

Lecture - 46
Basic Properties of L 1-Functions and Lebesgue's Dominated Convergence Theorem

(Refer Slide Time: 00:13)



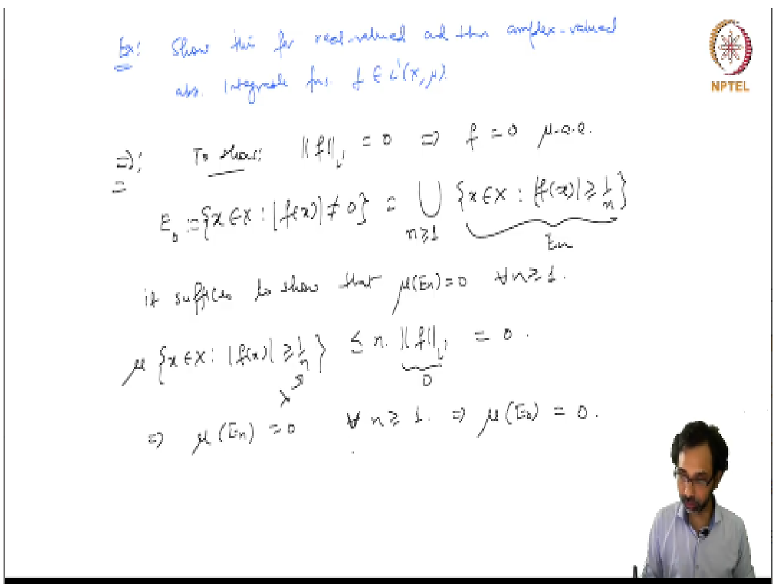
So, let us prove the first part and let us prove the reverse implication first which is that f = 0 mu almost everywhere implies that the L 1 norm is equal to 0. So, first note that f = 0 is equivalent to saying that mod f = 0 for a any point x. So, let me take x here. So, then it is enough to show this property for unsigned measurable functions. So, it suffices to show this for unsigned measurable functions because once you show it, show it, show it for unsigned measurable functions.

You can do it for real valued functions and then for complex valued functions as well. So, let us show this first for unsigned measurable functions. So, if f = 0 for mu almost everywhere and so, first we suppose that f is simple. So, we can do this easy case first. So, let me write f as a finite linear combination of indicate indicator functions for measurable sets E i. So, these are measurable sets.

So, then f = 0 mu almost everywhere implies that for each i, either alpha i; the value of f is equal to 0 or the measure of E i is equal to 0. So, this implies that the simple integral of f which is the same as the limit integral of f. This is equal to the sum alpha i mu E i and this is going to be 0. So, it holds for the case when f is simple. Now, if you take any positive simple function which is bounded above point wise by f.

Then f = 0 mu almost everywhere implies that s = 0 mu almost everywhere which means that the simple integral of s d mu. This is 0 by what we have just shown. And so, the supremum of all such simple functions is going to be 0 and this is nothing but the integral of f. So, we have shown that first unsigned measurable functions when f is 0 almost everywhere then the L 1 norm is going to be 0.

(Refer Slide Time: 03:52)



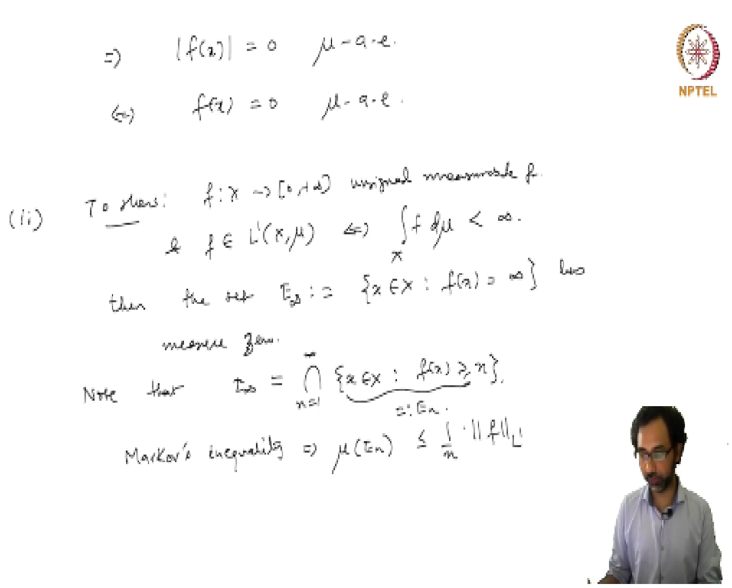
So, now I leave it as an exercise to show this for as an exercise. Show this for first real valued and then complex valued measurable absolutely integrable functions. This is x; f is in L 1 x mu by breaking these real valued functions in positive and negative parts and using the unsigned case and then using for the complex case, breaking it, breaking into real and imaginary parts and then using the real case. So, I leave it as an exercise.

Now, to prove the forward implication which is that if f has the L 1 norm 0 then f = 0 mu almost everywhere. So, note that the set. So, the set of points for which f x not equal to 0. It is a union a countable union of these sets f. So, let me write again mod f because if we show that mod f x equal not equal to 0 has measure 0 then f = 0 also has measure 0. So, mod f is greater than or equal to 1 over n.

And so, if we write these as E n, it suffices to show that the measure of E n = 0 for all n greater than or equal to 1. So, let us see what are these measures. x in x says that mod f x greater than or equal to 1 by n and this by the mark of inequality if you take lambda to be 1 by n then you have 1 over lambda. So, n 1 over 1 over n. So, n times the L 1 norm of f and this is 0. So, this is 0.

So, this implies that mu of E n = 0 for all n greater than or equal to 1 and this shows that this exceptional set where the modulus is not equal to 0.

(Refer Slide Time: 06:56)

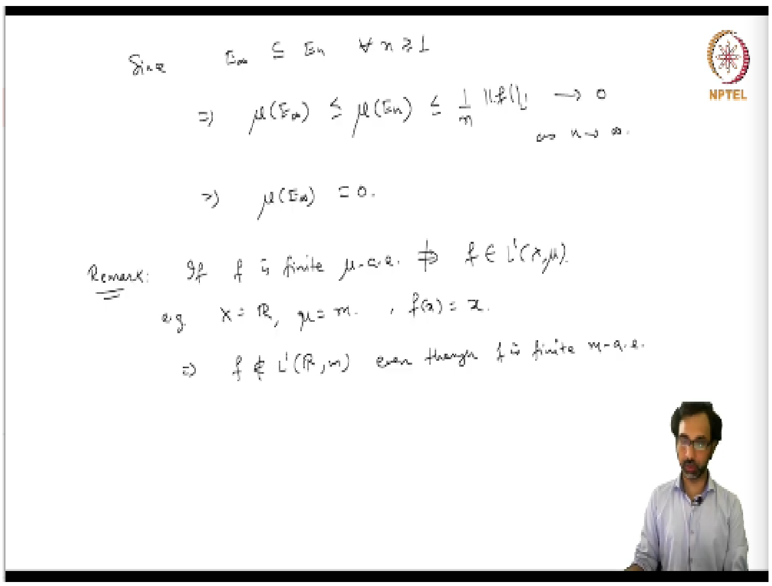


So, let me write this as E, E 0. So, this implies that the measure of E 0 is equal to 0 which means that this implies that the set of points for which mod f x is equal to 0 or rather mod f x = 0 mu almost everywhere and this is equivalent to saying that f x = 0 mu almost everywhere. So, we have shown that if L 1 norm is 0 then f = 0 mu almost everywhere. So, this shows the first part and now let us see what was the second part.

So, if f is an unsigned measurable function then f is finite mu almost everywhere. So, for the second part we have to show that if f is an unsigned measurable function and f belongs to the L 1 absolutely integrable function which is the same as saying that this integral d mu is finite. So, then we have to show that the set E infinity which is the set of points such that f x is equal to infinity has measure 0.

So, notice first that this set E infinity can be written as an intersection over all positive integers of the sets x in x says that f x is greater than or equal to n. And if these sets are, I denote E n then by Markov's inequality implies that the measure of the set E n is less than or equal to 1 over n times the L 1 norm of f.

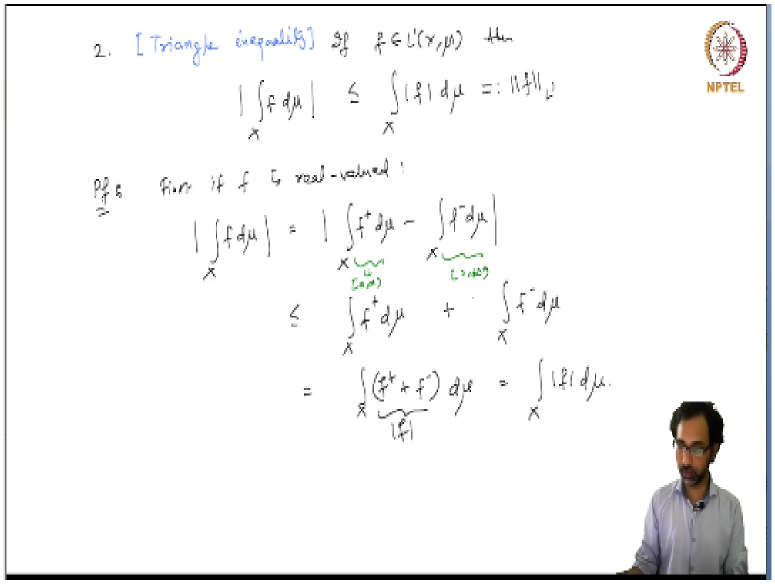
(Refer Slide Time: 09:23)



And since E infinity is a subset of E n for all n greater than equal to 1. This implies that the measure of E infinity is less than or equal to the measure of E n and this is less than or equal to 1 over n times the L 1 norm of f and this goes to 0 as n tends to infinity. So, this means that the measure of E infinity is 0. So, we have proved these 2 properties in the corollary. As a remark, I note that if f is finite mu almost everywhere then this does not imply that f belongs to L 1 x mu in general.

So, for an example you can take x to be R and mu to be the Lebesgue measure on R and f to be the identity function which gives you x. So, we know that this function is unbounded. And so, f does not belong to L 1 of R m. Even though f is finite actually everywhere here but in even if you leave out any null set, we also have f is finite almost everywhere for the Lebesgue measure. So, the converse for the corollary the second part of the corollary does not hold in general.

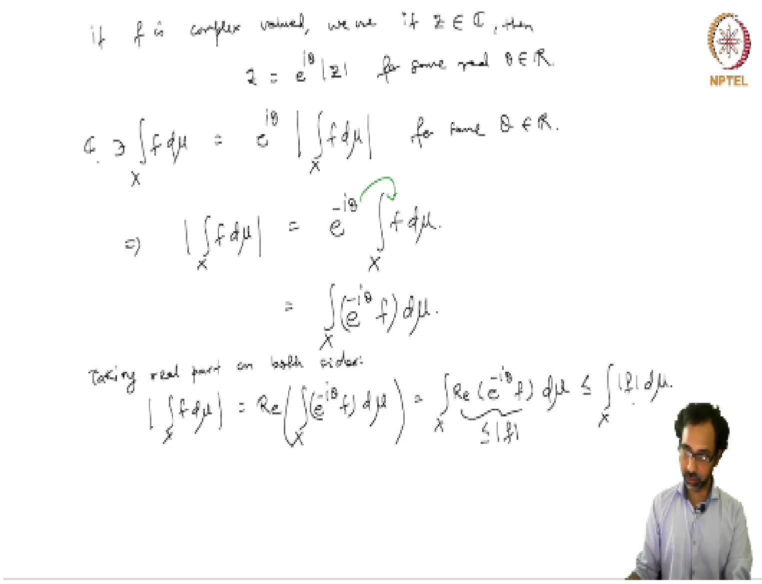
(Refer Slide Time: 11:17)



Now, let us look at another important inequality called the triangle inequality and it says that if f is an L 1 function then the modulus of the integral of f is less than or equal to integral of the modulus which is nothing but the L 1 norm. So, the L 1 norm bounds the modulus of the complex valued integral of f. So, let us see a proof. So, first if f is real valued then the modulus is nothing but the modulus of the positive part minus the negative part and these are 2 complex numbers.

So, this can be written as less than or equal to integral f + d mu plus the modulus of f - d mu but note that these 2 are already positive. So, you can remove the modulus because they are both positive. And so, this is equal to f+ + f- d mu and this is nothing but mod f. So, this is equal to mod f d mu. So, if f is real valued then it is then it follows from the usual triangle inequality for complex numbers. In fact, real numbers here because these are real numbers; positive real numbers.

(Refer Slide Time: 13:36)

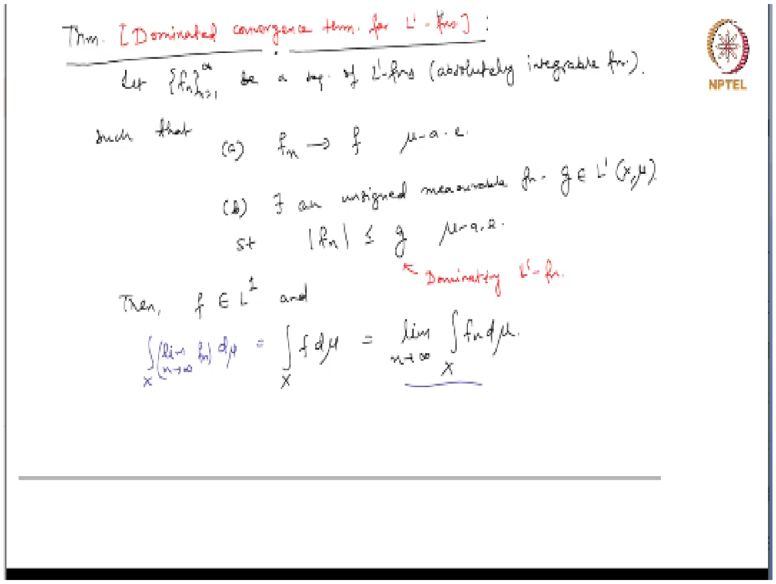


So, we have proved this for the case when f is real valued. Now; if f is complex valued, we use that if z is a complex number then z can be written as e to the i theta modulus of z for some real number theta. So, similarly we can write the integral f d mu as e to the power i theta modulus of f d mu because this belongs to the complex numbers. So, this is for some theta in R. So, this implies that the modulus of f d mu is equal to e to the power -i theta integral x f d mu and by using the linearity property using the for complex scalars.

Now, this is equal to, you can take this complex killer inside the integral. So, you get e to the power -i theta f of d mu and now you can take the real parts on both sides. So, taking real part on both sides. So, on the left hand side, it is already a real number because you have taken the modulus. So, it does not change. And on the right side, we have the real part of e to the power -i theta f b mu but the way we have defined complex integrals the real part of the complex integral is the integral of the real part.

So, this is nothing but the real part of e to the power -i theta f d mu and now this is less than or equal to mod f. So, this is less than or equal to modulus of f. So, we have shown that this also holds for complex valued L 1 functions.

(Refer Slide Time: 16:04)



So, now we come to a very important theorem called the dominated convergence theorem for L 1 functions or absolutely integrable functions. So, we have seen that we can use the monotone convergence theorem for unsigned measurable functions to interchange the limit and integral and the dominated convergence theorem gives you a sufficient criteria when you can interchange the limit and integral for complex valued L 1 functions.

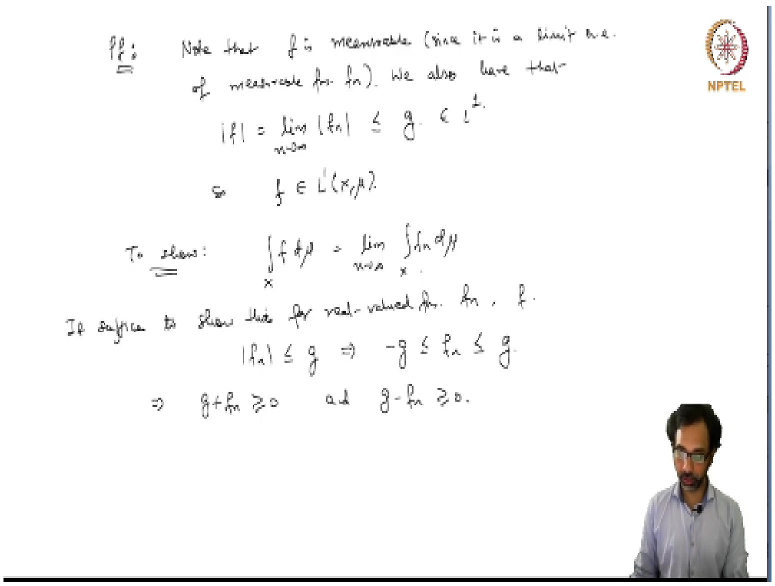
So, let us see the statement of the theorem. So, let f n n = 1 to infinity be a sequence of L 1 functions. So, here I am abbreviating absolutely integrable functions as L 1 functions absolutely integrable functions. This is L 1 functions in short. So, if you take a sequence of L 1 functions such that first condition is that f n converges to f, a function f, mu almost everywhere.

And secondly that there exists an unsigned measurable function g which is also absolutely integrable or integrable such that the all these f n's the modulus of all these f n's is bounded above by g again mu almost everywhere. So, then the statement of the theorem says that then we can interchange the limit and the integral sign which means that the integral. So, first that f is in L 1 and the integral of the complex integral of f is equal to the limit as n tends to infinity of the integrals of f n.

And here again this is a interchange of limit and integration because on the left hand side you have the limit inside and here you have the limit outside the integral on the right hand side. So, this is called the dominated convergence theorem because we have a dominating function g dominating L 1 function g which bounds all these f n's and this guarantees that your limit is going to be L 1 and you can write the limit the integral of the limit is as the limit of the integrals.

So, here we use the technical condition that our measure space is complete. So, let me just write it here. So, we let x B mu be a complete measure space. So, this holds for complete measure space because we have seen that for complete measure spaces a sequence of measurable functions converging almost everywhere to another function gives you a measurable function.

(Refer Slide Time: 19:58)

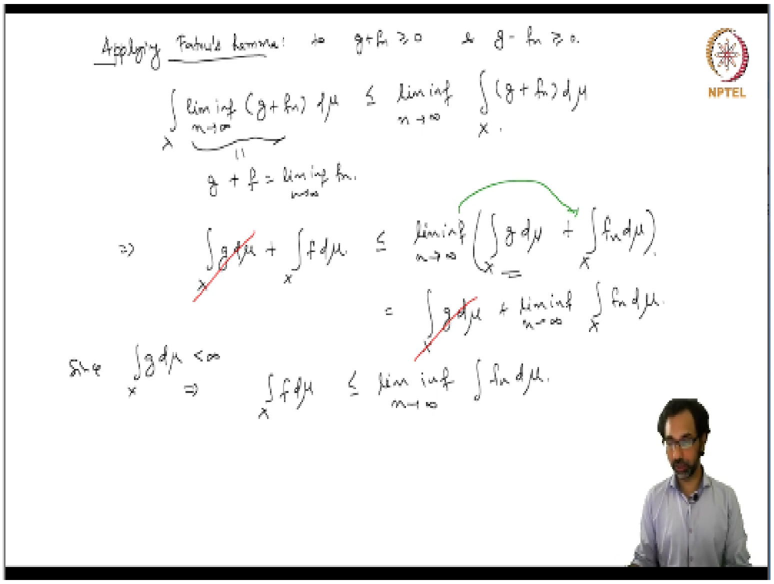


So, we need this completeness business but once we have that it is not very difficult to establish the proof. So, first note that f is measurable since it is a limit of measurable functions. It is a limit almost everywhere of measurable functions f n. Secondly, we also have the modulus of f is equal to the limit as n goes to infinity of the modulus of f n and this is bounded above by this function g and this is an L 1 function.

So, f is in fact an L 1 function. So, now we are only left to show that the integral of f d mu is the limit of the integrals of f n d mu. So, first note that it suffices to show this for real valued functions only by taking comp real and imaginary parts for complex valued functions. So, it suffices to show this for real valued functions f n and f. So, we assume that all the f n's are real valued and all and the limit f is of course again real valued.

So, then we have that modulus of f n is less than or equal to g. This is given this implies that -g less than or equal to f n is less than or equal to g. So, f n lies between -g and g. And this implies that first that g + f n is positive and g - fn is also a positive function.

(Refer Slide Time: 22:23)

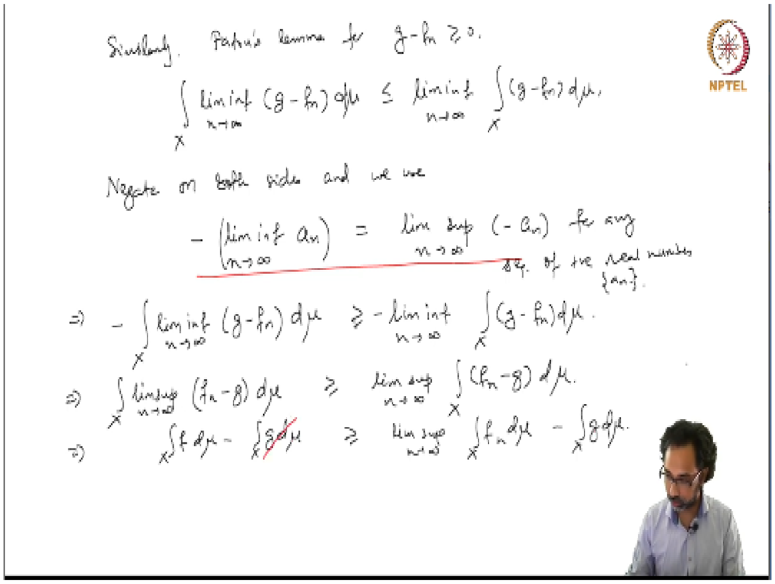


So, now we are going to apply Fatou's lemma, applying Fatou's lemma. So, remember that Fatou's lemma was only applicable for unsigned measurable functions. So, we are going to apply it for both these functions g + n to g + n f n and g – f n. So, first if we take g + f n, we will get integral lim inf of g + f n d mu n goes to infinity is less than or equal to lim inf as n goes to infinity integral of g + f n d mu.

Now, this is nothing but g + f because this is the lim inf and lim sup of f n both. So, in particular it is equal to the lim sup; f is equal to the lim sup. So, the lim inf of g + f n is equal to g + f. So, this implies that integral g d mu + integral f d mu by using linearity. I am writing it as 2 integrals g d mu + f d mu. And this is less than or equal to the lim inf as n goes to infinity integral g d mu + integral f n d mu but this first one does not have any dependence on n.

So, you can take this lim inf inside. So, this is equal to integral g d mu + lim inf f n tends to infinity f n d mu and now we can cancel these two g d mu's integral because these are finite. So, since integral g d mu is finite, this implies that integral of f d mu is less than or equal to lim inf n tends to infinity integral f n d mu.

(Refer Slide Time: 25:02)



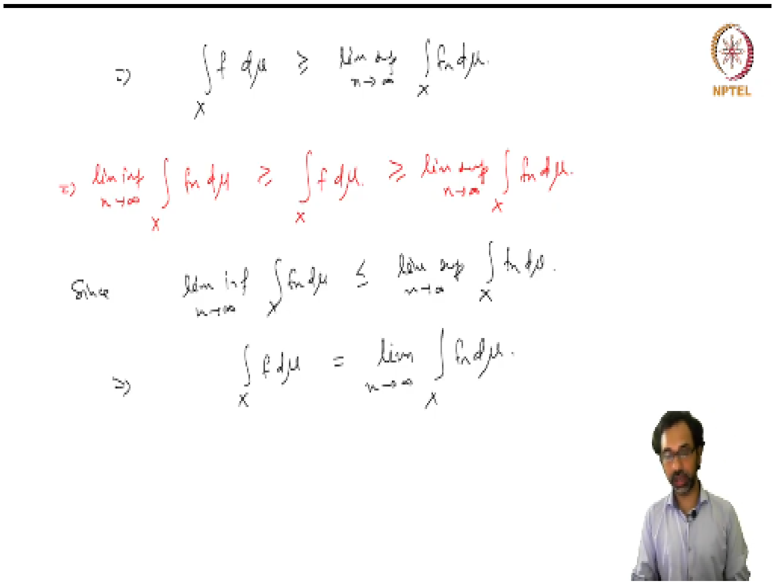
Similarly, we can use it. We can apply Fatou's lemma for Fatou's lemma for g – f n which is again a positive sequence of functions and for this, we again have lim inf n goes to infinity g - f n d mu is less than or equal to the lim inf as n tends to infinity integral g - f n d mu. And now we take the negation on both sides.

So, negate on both sides and we use the fact that lim inf n tends to infinity with a negative sign of some sequence a n is equal to lim sup n tends to infinity of -n for any sequence of real of positive real numbers of positive real numbers a n. So, here we will get minus integral of lin inf n tends to infinity g - f n d mu is greater than or equal to minus lim inf n tends to infinity integral g - f n d mu.

And so, now we can use this fact here of about lim sup and lim inf when you take the negation. So, this is nothing but integral lim sup n tends to infinity minus or rather f n - g f n - g d mu and on the right hand side, you will get lim sup n tends to infinity integral x over x f n - g d mu but on the left hand side, the lim sup of f n's is simply f. So, you get integral f d mu minus integral g d mu.

And on the right hand side, you get lim sup n tends to infinity integral of f n d mu minus integral of g d mu but now this function g is an L 1 function. So, these terms are finite the integrals are finite. So, they cancel out.

(Refer Slide Time: 28:27)



And this implies that integral over x f d mu is greater than or equal to the lim sup of n tends to infinity of f n d mu. So, in total, we get that the lim inf of f n d mu n tends to infinity is greater than or equal to the integral f d mu which is greater than or equal to lim sup n tends to infinity f n d mu and because we always have since lim inf n tends to infinity f n d mu integral is less than or equal to the lim sup f n d mu.

This implies what we need which is that the integral of f d mu is equal to the lim sup or the lim inf and the limit when they are all the same, we just write limit lim of f n d mu. And this is the proof of the dominated convergence theorem.