Measure Theory

Prof. Indrava Roy
Department of Mathematics
Institute of Mathematical Science

Lecture - 45
Lebesgue Integral for Complex and Real Measurable Functions: The Space of L 1 Functions

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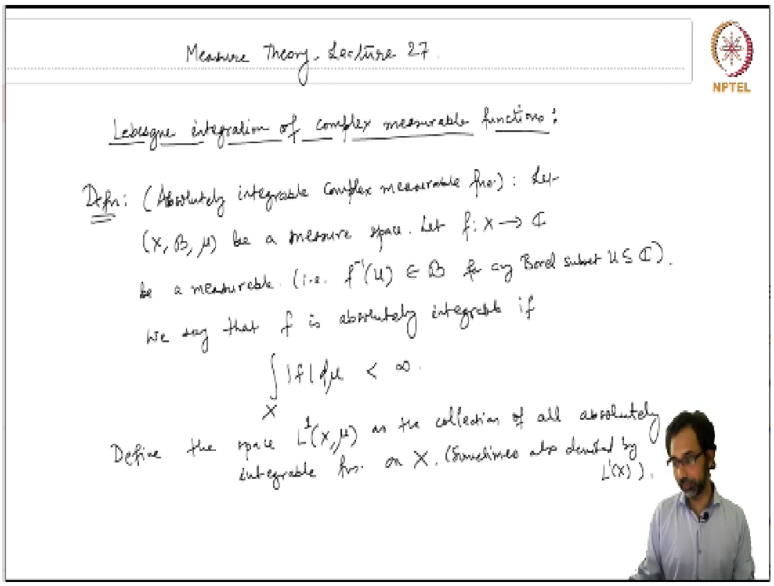

Recall that until now we have looked at the Lebesgue integration formula for simple measurable functions and unsigned measurable functions. So, in this lecture, we will look at the Lebesgue integration formula for complex measurable functions. So, let us give the definition for absolutely integral complex measurable functions. So, let x B mu be a measure space and let f which is a map from x to the complex numbers be a measurable function.

Recall that this means that every it, f pulls back Borel sets to measurable sets. So, this means that f inverse u belongs to this sigma algebra B for any Borel subset u of c. So, the Borel subsets are elements of the Borel sigma algebra generated by the open subsets of c. So, we call a complex measurable function, complex function measurable if this condition is satisfied. And so, for a complex measurable function f.

We say that f is absolutely integral if the integral over x of the modulus of f d mu. This is finite. So, this is a complex number and we say that when it is a finite complex number, this is f, is called absolutely integral. And so, we can define the space L 1 of x mu. This is the collection as the collection of all absolutely integral functions on x. So, notice that here L 1 x mu has information both about the measure mu as well as the sigma algebra B.

So, we do not usually write L 1 x be mu. We suppress the notation somewhat and just write L 1 x comma mu and even sometimes write sometimes also denoted by simply L 1 of x. So, when the measure is clear. We do not have to repeat it, will repeat the notation for mu always and we simply write L 1 of x.

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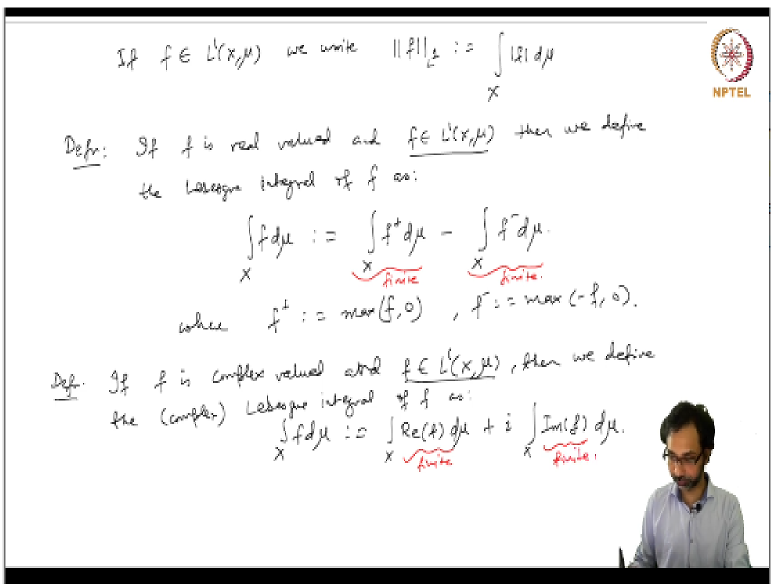

Now; the quantity for an absolutely integral function. So, if f is in L 1 x mu, we write the L 1 norm of f as this finite quantity of the integral of mod f d mu. Further, we can make the following definitions that if f is real valued and f belongs to L 1 x mu. Then we define the Lebesgue integral of f as integral of f d mu. So, notice that we only worked with mod f which was an unsigned measurable function but here we have replaced it with f.

So, f is real valued but we can break it up into its positive and negative parts and then we are back to the unsigned case again. So, we can write it as f+ d mu - f- d mu. So, where f+ is the maximum of f and 0 and f- is the maximum of minus f and 0. So, these are the positive and negative parts of the real valued function f. And we define the real living integral for this real value function as the difference of these two unsigned these two integrals for the unsigned measurable functions f+ and f-.

Similarly, if f is complex valued and f is in L 1 x mu then we define the complex Lebesgue integral of f as this is the integral of f d mu and it is; now we can break it up into its real and imaginary parts which are both real valued and so, we have already defined what is a real valued what is the integral for a real valued function. So, now we can use this. So, real part of f d mu + i times imaginary part of f d mu.

So, first we define it for we define the Lebesgue integral for unsigned measurable functions and then we define it for real valued absolutely integral functions and then real complex valued absolutely integral functions. So, we require this condition of absolute integrability because when we have absolutely integral functions. Then for example, if it is real valued then both these integrals these are finite.

So, this is a finite integral and this is also a finite integral and so the difference is also finite. So, this is only when f is already absolutely integral. Similarly, if f is absolutely integral and complex valued then both these integrals of the real and imaginary parts are finite.

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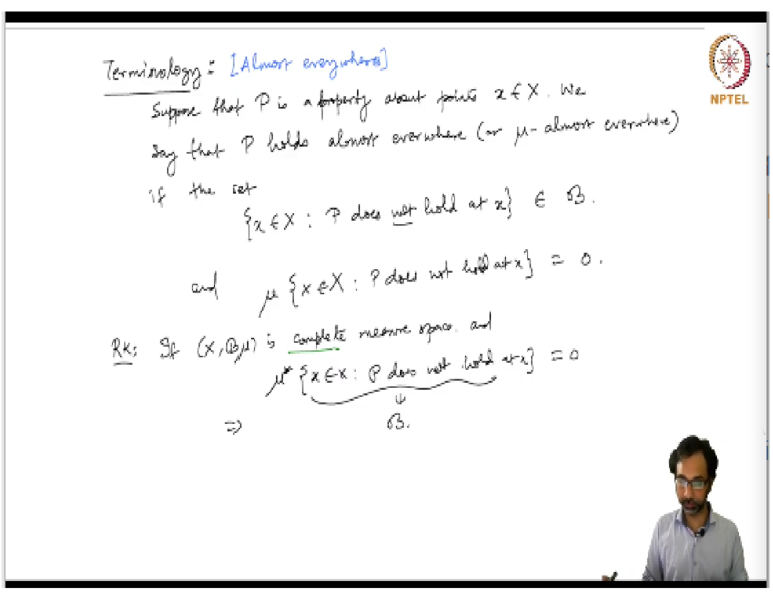

So, for this definition or terminology, suppose that P is a property about points of x. So, we say that P holds almost everywhere, almost everywhere or mu almost everywhere sometimes. If we want to emphasize the measure then we say it is mu almost everywhere. If the set of points; so, let us take the set of points such that P does not hold at x, does not hold at x. This is a measurable set and the measure of the set is 0.

So, this set of points such that P does not hold, hold at x this measure is 0. We will often work with complete measure spaces. So, if we can, we can also rewrite it as the outer measure of this is 0 then it will automatically hold that this is a measurable set if x is complete. So, let me write it as a remark. So, if x B mu is complete, is a complete measure space; we call that a complete measure space is one which for which all subsets of null sets are also measurable.

So, if x B mu is a complete measure space and the outer measure of this set of points on which P does not hold = 0. Then this implies that this set belongs to the sigma algebra B. So, we will often work when we have to assume that x B mu is a complete measure space.

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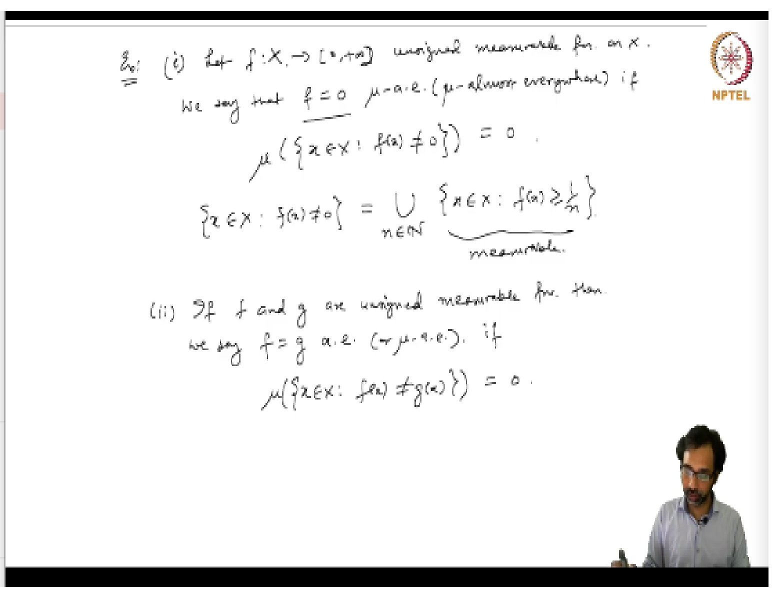

So, for example, so let us take an example of a property which holds everywhere. So, we hold almost everywhere. So, let f be a an unsigned measurable function on x. So, we say that for example that f = 0 mu a e in short. So, this is mu almost everywhere, mu almost everywhere. I f the measure of the set of points such that f x is not equal to 0. This is 0. Now, note that this is a actually automatically a measurable set because we can write this set of points for which f x not equal to 0.

This is equal to the union over natural numbers of the sets x in x such that f x is greater than or equal to 1 over n. So, when you take the union, you get this set of points for which f x is not equal to 0 and of course, these are measurable because f is a measurable function. So, in this case we automatically have that your set of points for which this property does not hold which is f = 0 does not hold; is already measurable and we say that f = 0 almost everywhere if this set has measure 0.

Similarly, if f and g are unsigned measurable functions then we say that f = g almost everywhere or mu almost everywhere. If the measure of the set of points such that f x is not equal to g x is equal to 0. So, we see that we can take various properties of our functions or even the space x itself. And we have this notion of almost everywhere when the complement of that property has measures 0 or the negation of that property has measures 0.

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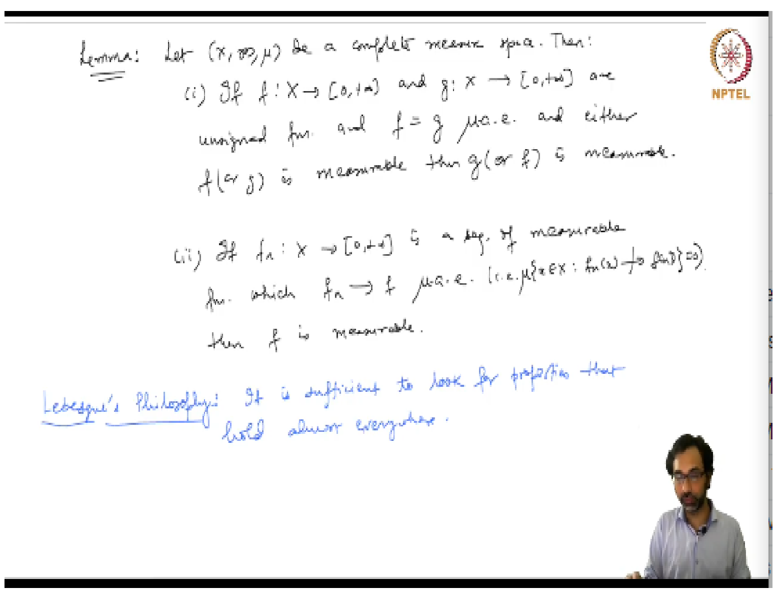

So, we have actually a nice result about almost measurable functions. So, let x B mu be a complete measure space and so, then let me write 2 properties actually. So, first one is that if f and g are unsigned functions are unsigned but not necessarily measurable but we suppose that one of them is measurable. And either so, first let me write that f = g mu almost everywhere and either f or say g is measurable.

Then the other one is also measurable. Then g or f is measurable. Similarly, we say that if f n is a sequence of measurable functions for which f n converges to f for almost every x which is mu almost everywhere. So, this means that f n might not converge to f at points whose measure is 0. So, this means that the set of points says that f n x does not converge to f x as measure equal to 0. So, if you have a sequence of measurable functions which converges almost everywhere to a function f then f is measurable.

So, this is a very nice property of Lebesgue's measure theory that we can leave out a set of measures 0 and we can retain many of the important properties of measurability and absolute integrability as well. So, this is Lebesgue's philosophy. So, it says that in informal terms, it is sufficient to look for properties that hold almost everywhere for all practical purposes in Lebesgue's measure theory.

So, measurability and absolute integrability, we can also look at point wise convergence almost everywhere and so on. So, this gives us a lot of leverage and a lot of free space to work with measurable and absolutely integral functions.

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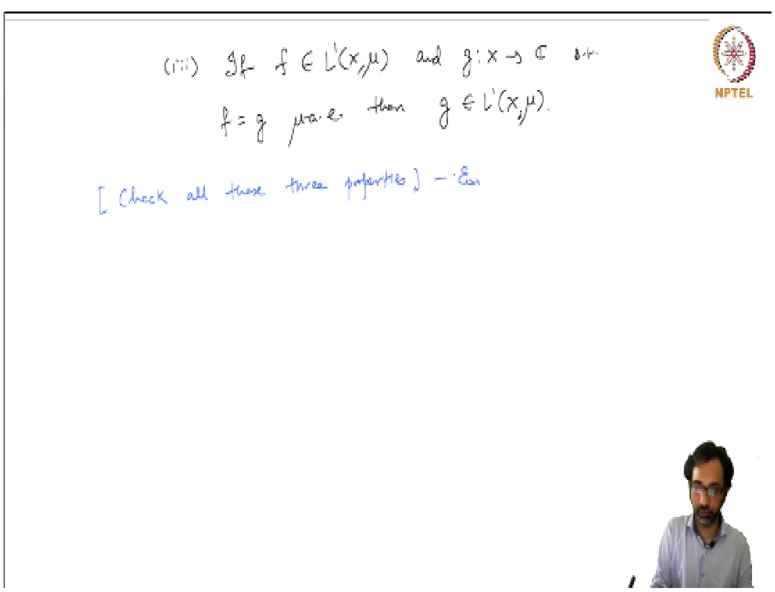

So, another property here is that if you replace. So, let me write it down. So, let me write the third property in this lemma is that if f is an absolutely integral complex valued function and g is another function complex valued. Such that f = g mu almost everywhere then g is also in L 1. So, we have defined an absolutely integral function as one whose modulus has a finite Lebesgue integral as an unsigned measurable function.

And if you take any other g which is which agrees with f for a large set which means that the complement of that set has measure 0. Then g also belongs to this set of absolutely integral functions. So, I am going to leave that leave all these 3 as an exercise. So, check all these 3 properties. So, this is an exercise for you. So, just by following the definition one can show these properties quite easily.

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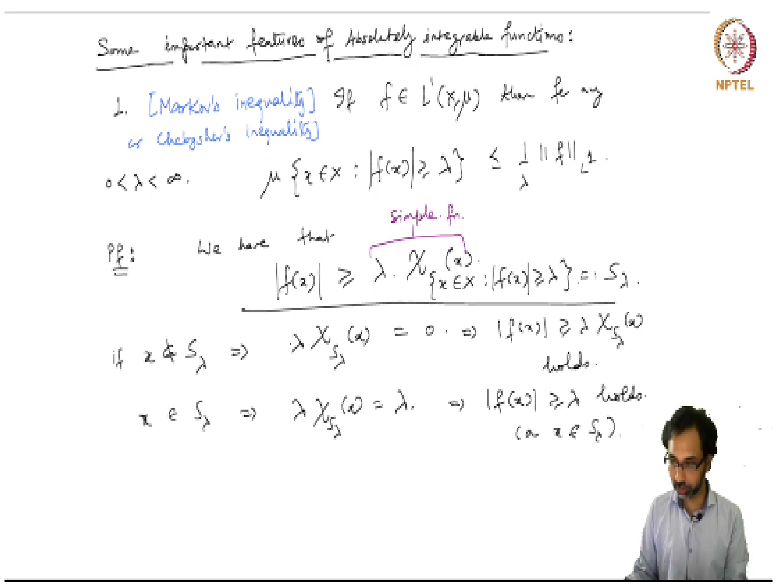

Now, let us look at some important properties of absolutely integral functions. So, the first one is called Markov's inequality. It is also sometimes called Chebyshev's inequality. So, this says that if f is L 1 function. Then we have the following inequality which is that the measure of the set of points such that. So, this is for any finite number lambda finite positive number lambda between 0 and infinity.

So, the set of points for which f x is greater than or equal to lambda, is bounded above by 1 over lambda with a norm L 1 norm of f. So, it is actually quite easy to show. So, let me write the proof. So, we know that. So, we have that f x is greater than or equal to lambda times the indicator function of the set x belong to x as that f x greater than or equal to lambda. So, this is a easy inequality to verify because if f x is a.

So, this should be mod f. Sorry. So, we have this inequality with the mod. So, if f is unsigned then we can remove of the mod but in general we have an complex valued function. So, we take the modulus of f x. So, the modulus of f x is greater than or equal to lambda times the indicator function of this set where the modulus is greater than or equal to lambda. This is because if x does not belong to this set.

Then the indicator function will give you 0 and you will only get. So, there is an x here. You will only get 0 on the right hand side. So, you will get mod f x is greater than equal to 0 which is always true and if you have x belonging to this set then you will have a 1 and you will get mod f x greater than or equal to lambda. So, let me write so if x does not. So, let me write this set as s lambda.

So, if x does not belong to s lambda this implies that modulus of f x, is so lambda times chi s lambda of x is equal to 0. So, this implies that mod f x greater than or equal to lambda chi s lambda x holds and if x belongs to s lambda. This implies that lambda chi s lambda x is equal to lambda simply. This implies that mod f x is greater than or equal to lambda holds because x belongs to s lambda.

So, in both cases we see that this inequality holds and now we can use the monotony scenario property. So, note that this function on the right hand side. This is a simple function now; simple measurable function. And so, we can integrate on both sides.

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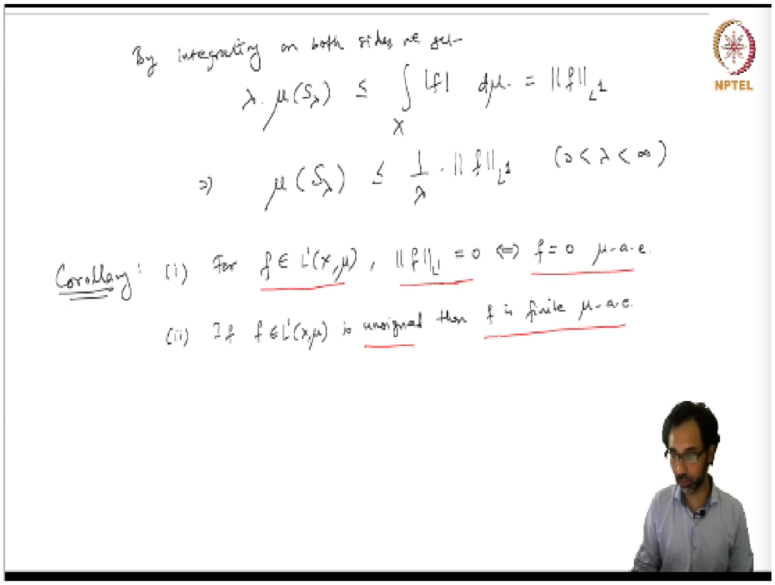

So, we get so by integrating on both sides. We get lambda times the measure of this set s lambda is less than or equal to the integral x mod f x d mu mod f d mu and this is nothing but the L 1 norm of f L 1. And so, the measure of this set s lambda is bounded above by 1 over lambda and times the L 1 norm of f and note that lambda is between 0 and infinity. So, we can divide on both sides by lambda and we get this inequality.

So, this is Markov's inequality and as corollaries, two important corollaries. So, the first one says that if f is an absolute integral function and the norm the L 1 norm of f is 0 then f = 0 mu almost everywhere and vice versa which says that if f = 0 mu almost everywhere; then also the L1 norm is 0. So, this is an if and only if condition. So, it says that if L 1 norm is 0 if and only if f = 0. The second conditions in the property says that if f is a L 1 function which is an unsigned measurable function then f is finite almost everywhere.

So, remember that for unsigned measurable functions, we allowed the value plus infinity and so, this says that f is the set of points for which f takes the value plus infinity has measure 0. So, let us prove this using Markov's inequality.