Measure Theory
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Lecture - 44
Fundamental Convergence Theorems in Lebesgue Integration, Monotone Convergence Theorem, Tonelli's Theorem and Fatou's Lemma

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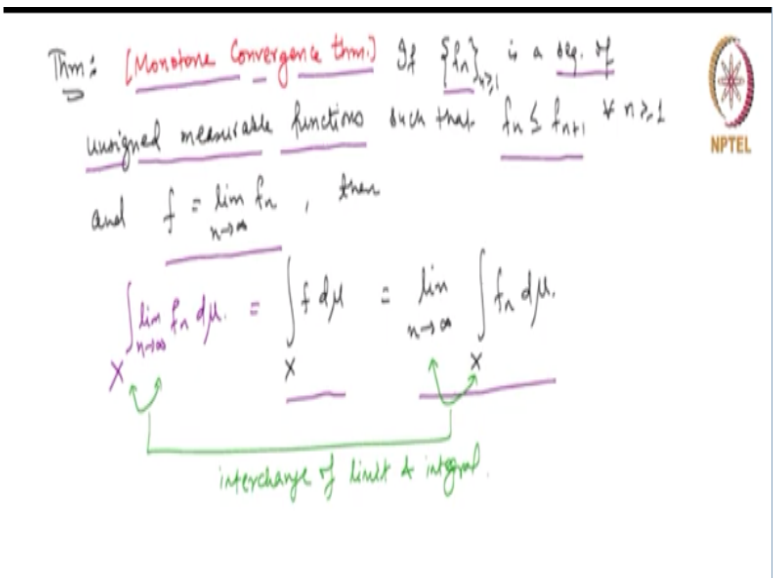

Now we come to an extremely important property of the Lebesgue integral, which is the monotone convergence theorem and it says that if fn is a sequence of unsigned measurable functions such that fn is less than or equal to fn + 1. So it is an increasing sequence or non-decreasing sequence of unsigned measurable functions and f is the pointwise limit of these functions fn, then the integral of f is equal to the limit of the integrals of the fn's.

So note that, on the left hand side we have the integral of the limit n tends to infinity, fn d mu and on the right hand side, we have the limit n tends to infinity integral fn d mu. So one can view it as the interchange of limit and integral signs, which is allowed when you have a sequence of non-decreasing unsigned measurable functions converging pointwise to a function f. So this is an interchange of limits and integration signs. So let us look at the proof for this monotone convergence theorem.

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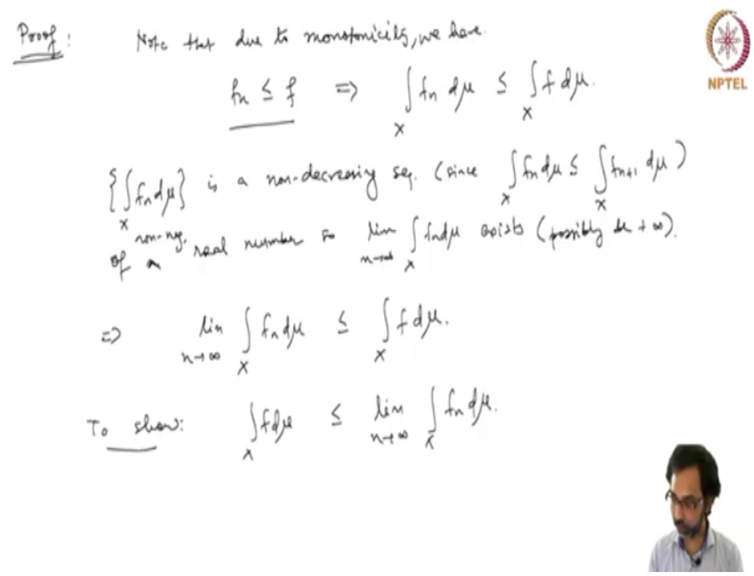

So first note that due to monotonicity, we have that: since fn is less than or equal to f, this implies that integral fn d mu is less than or equal to integral xf d mu. So of course if fn is a non-decreasing sequence converging to a function f, then the limit bounds all the fns. So we have fn is less than or equal to f and this implies by the monotonicity property that the integrals also satisfy this inequality. Now this sequence of integrals fn d mu is a non-decreasing sequence.

Since we also have that fn d mu is less than or equal to fn + 1 xd mu; of real numbers of course, of positive real numbers, non-negative real numbers. So the limit exists. So limit n tends to infinity integral xfn d mu exists in the extended real numbers, which so, it could possibly be plus infinity as well, but since it is a non-decreasing sequence, the limit must exist. Whether it is a finite real number or plus infinity.

So this implies that the limit of n tends to infinity integral xfn d mu is also bounded above by f d mu. So now it suffices to prove the reverse inequality. So to show that integral xf d mu is less than or equal to the limit n tends to infinity d fn mu.

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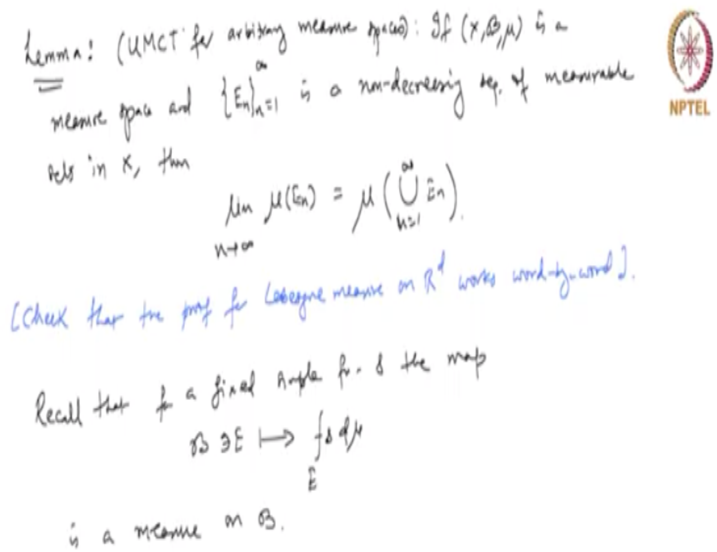

So to show this reverse inequality, we will use the following lemma, which is the upward monotone convergence theorem for arbitrary measure spaces and it says that if xd mu is a measure space and En, n = 1 to infinity, is a non-decreasing sequence of measurable sets in x, then the limit of n tends to infinity measure of mu En is precisely equal to the measure of the union n = 1 to infinity En.

So we have seen this result for the Lebesgue measures, but just check here, check that the proof that we used for Lebesgue measure works word by word in this abstract setting as well. So check that the proof for Lebesgue measure on Rd works word by word. So it is almost trivial generalization for the upward monotone convergence theorem for the Lebesgue measure. So we will use and we will also use the fact that we have proved.

Recall that for a fixed simple function s the map from E to for a measurable set E and B, which are signs, this set E the real number given by the simple integral over E. So we have defined the simple integral over any measurable set E and so we can associate to any measurable set E, this number and this is a measure on B. So I left it as an exercise before and we are going to use these two results.

The first is that the upward monotone convergence theorem holds and the second is that this assignment of the simple integral or fixed simple function s to any measurable set; this gives you a measure on the sigma algebra B.

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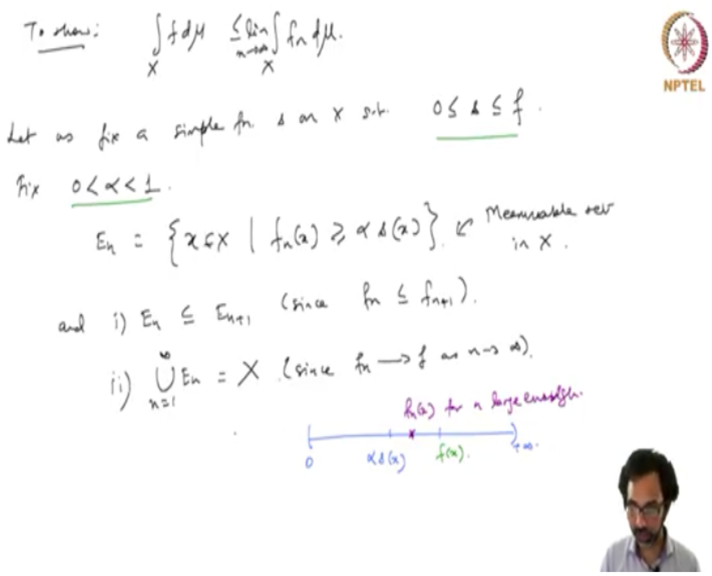

So let us see how to prove the reverse inequality with these two properties. So we have to show that the integral of f d mu is less than or equal to the integral of fn d mu limit n tends to infinity. So let us fix a simple function s on x such that we have 0 less than or equal to s less than or equal to f. So we are going to use the fact that the Lebesgue integral of f is defined using simple functions, which are bounded above pointwise by f.

So let us fix any such simple function s, which is bounded above by f. Also, fix a constant alpha, which is strictly between 0 and 1. Now if you define En as the set of points in x, such that fnx is greater than or equal to alpha times sx. Now this is a measurable set in x and we also have that En is a subset of En + 1, since fn is less than or equal to fn + 1. So this is automatic. So this is the first property that En satisfies and the second property is that the union of all these En, n = 1 to infinity, this is actually equal to the whole space x.

And this is due to the fact that fn converges to f as n tends to infinity, because if you see on the positive or non-negative real line, 0 to infinity, so let us say that fx is some finite value and because we have chosen s to be less than or equal to f and alpha to be strictly less than 1, so alpha sx will lie strictly below fx and so using the property of the limit, we can find fn x lie between these two points for n large enough, because fn converges pointwise to f. So this implies that the union of all these E n s is the whole space x.

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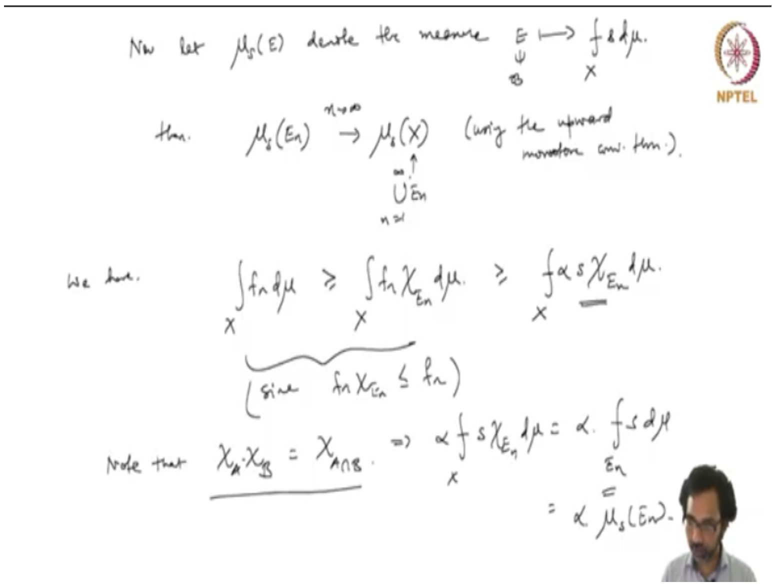

Now let mu s of E denote the measure obtained by associating with the measurable set E, thus integral of sd mu. So if you denote mu s, the measure, then we have that mu s of En as tends to infinity converges to mu s of x as using upward monotone convergence theorem that we just saw. This is because x is the union of this Ens. So therefore, we have first that the integral over x of fn d mu is greater than or equal to integral over x of fn chi E n d mu.

This is again because fn chi En is pointwise bounded above by fn, so we have just used monotonicity and then on this set En, we have that fn is greater than or equal to alpha times sx. So this is greater than or equal to alpha times sx or s chi E n d mu. Now this is a simple integral, because we have in our simple function s chi En, but note that chi A chi B, if you take the indicative functions of two sides A and B, and you multiply them, then this is the indicative function of the intersection.

So this implies that the integral alpha s chi En over xd mu is nothing but alpha of the integral s, simple integral over the set En, because when you are considering the simple integral over a measurable set, you just take intersections and so using this fact, we can easily prove that. When you put chi En and multiply with s, you get the simple integral over En itself. So now this is nothing but alpha times mu s of En and now we can take the limit on the right hand side.

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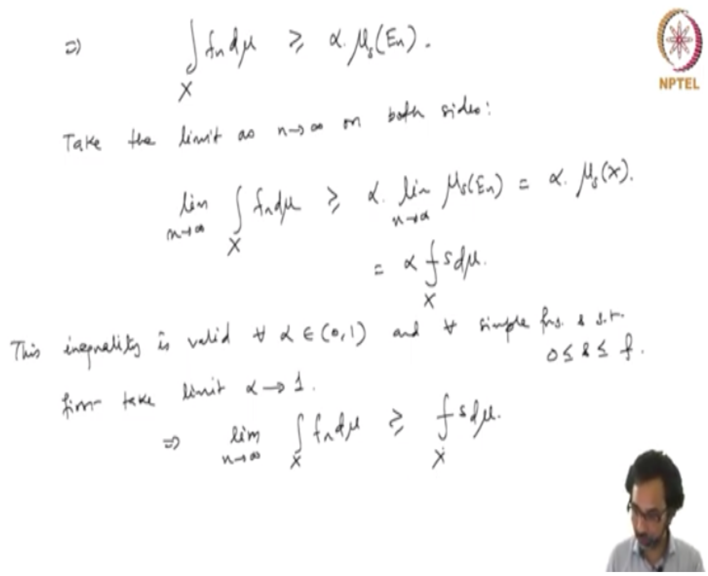

This implies that integral fn x d mu is greater than or equal to alpha times mu s of En. Now take the limit on both sides, limit as n tends to infinity on both sides, we get that the limit n tends to infinity xfn d mu is greater than or equal to alpha times limit n tends to infinity mu s En, but we have seen that this is nothing but alpha times mu s of the whole space x and this can be rewritten as alpha times x the simple integral of sd mu over the whole space x.

So now note that this is valid. This inequality is valid for all alpha strictly between 0 and 1 and for all simple functions s such that 0 less than or equal to s, less than or equal to f. So first, we can take the limit as alpha goes to the value 1, so this implies that the limit of n tends to infinity xfn d mu integral is greater than or equal to the value at alpha equal to 1, which is nothing but this simple integral and so therefore, since this is true for all s, we can also take the supremum on the right hand side of the inequality.

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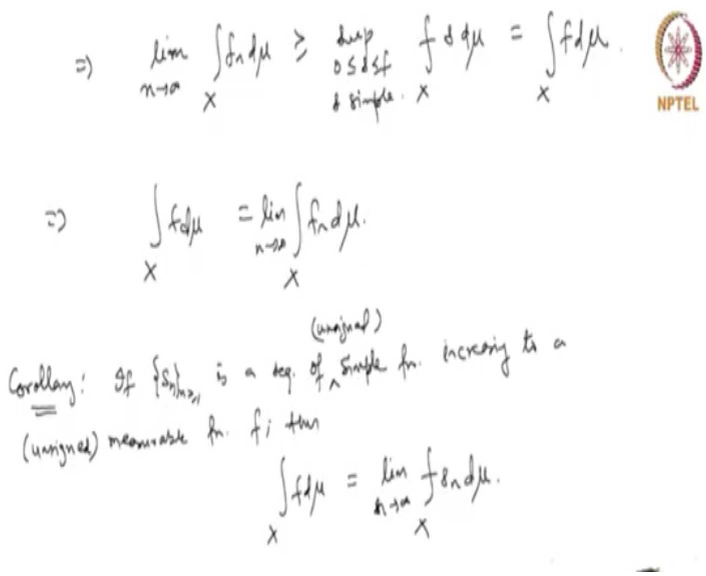

So this implies that the limit as n tends to infinity xfn d mu is greater than or equal to the supremum over the set of such simple functions, which are pointwise bounded by f and you take the simple integral of s and this is nothing but integral of f d mu. So we have shown the reverse inequality. This implies that integral of fd mu is equal to the limit n tends to infinity of fn d mu.

Now the monotone convergence theorem is a fundamental result in Lebesgue measure theory of integration and we will use it over and over again to derive many more interesting results as we go along, but the first corollary that interests us is the following. So if sn is the sequence of simple functions increasing to measurable function f. So all of these are unsigned and here also unsigned, then the integral of fd mu is equal to the limit of the s tends to infinity of the integrals of sn d mu.

And we have already seen that such as increasing sequence always exist for an unsigned measurable function. Therefore, we always have atleast one sequence which we can use to evaluate such an integral. So this gives the monotone convergence theorem gives you an immediate way to compute the Lebesgue integral in terms of increasing sequences of simple functions or even unsigned measurable functions.

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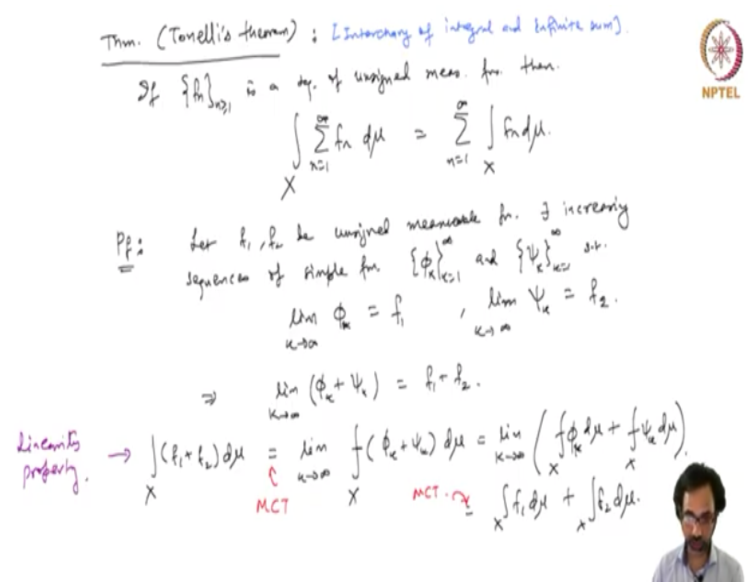

So the next result is called Tonelli's theorem, which is about interchange of infinite series, infinite sum and integral. So this is about interchange of integral and infinite sum. So it says that if fn is a sequence of unsigned measurable functions, then the integral of the series given by summing up all the fn d mu is equal to the infinite sum of the individual term by term integration. So this is about allowing term by term integration, just by taking sequence of unsigned measurable functions.

So let us see the proof. So first we have to show that this holds for finitely many functions fn. So we can start with two functions. So let f1, f2 be unsigned measurable functions and because they are unsigned measurable, there exists an increasing sequences of simple functions phi k, k = 1 to infinity and psi k, k = 1 to infinity such that the limit as k tends to infinity of the phi k = f1 and the limit as k tends to infinity of this psi k = f2.

So this implies that the limit as k tends to infinity of the sum psi k + phi k = f1 + f2. So now if you take the integral of the sum f1 + f2 d mu, then this is the limit as k tends to infinity of phi k + psi k. This is by the monotone convergence theorem, MCT in short and we have seen in the corollary that we can take any sequence of simple functions converging from below increasing sequence converging from below and then we will have this formula.

But now this is a simple integral, so we can use the linearity property for the simple integral to write this as the sum of phi kd mu + psi kd mu and then we can take the limit inside, because both limits exist. So this is equal to nothing but integral of f1d mu + f2d mu again by using MCT, because f1 d mu is the limit of the first term, integral of phi kd mu and the integral of f2 d mu is the limit of the integrals of psi k.

We see that the linearity property holds, this is the linearity property for Lebesgue integrals. We had only proved till now the linearity with respect to scalar multiplication, but now using MCT, we have also proved that linear with respect to addition of two functions.

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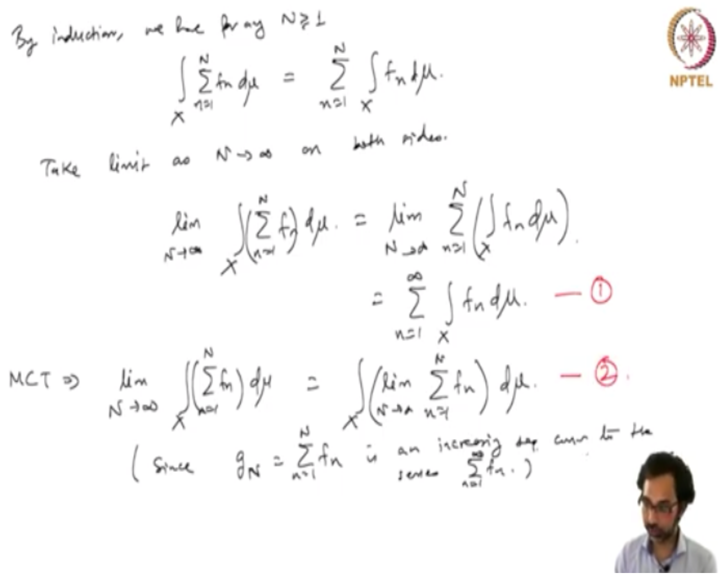

So this is a nice result and by induction we have for any n greater than or equal to 1. We have that N, here that the sum of n = 1 to N fn d mu is equal to the sum n = 1 to N of the individual integrals fn d mu. So now we can take the limit on both sides as N goes to infinity on both sides. So we get the limit as n tends to infinity x n = 1 to N fn d mu and this is equal to the limit as N tends to infinity n = 1 to N, integral over x of fn.

The right hand side is already something we want which is n = 1 to infinity integral fn d mu. On the other hand, we can use the MCT implies that the limit as n tends to infinity of this integrals sum n = 1 to N fn d mu is equal to the limit as n tends to infinity taken inside the integral n = 1 to N fn d mu, because since this sequence gN, which is the sum over n = 1 to N fn is an increasing sequence of measurable functions converging to the series n = 1 to infinity fn. So we can interchange the limits in this case using the monotone convergence theorem. So from first equation and second equation, we get the result.

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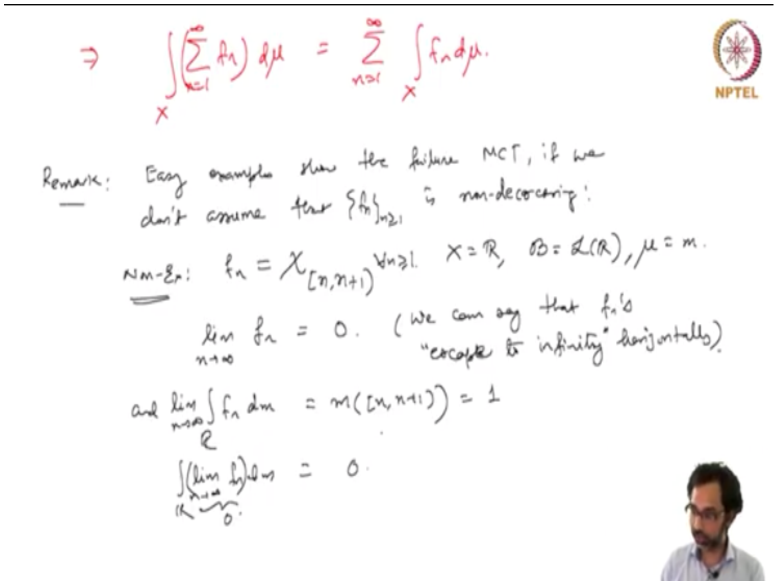

Which is that the integral of n = 1 to infinity fn d mu, the sum is equal to the sum n = 1 to infinity of the individual integrals of fn. So this is the Tonelli's theorem. Note that it was only allowed because all these fn are unsigned measurable functions, which makes this patched sums a non-decreasing sequence, so that we can apply the monotone convergence theorem. Now let me remark here that easy examples can show the failure of the monotone convergence theorem, if you do not assume that we have an increasing sequence of functions.

If we do not assume that the sequence of functions fn is increasing or non-decreasing. So let us seen an example. So we can take fn to be, so this is a rather a non-example. So we can take fn to be the indicative function for the interval n, n + 1. So here x is R and you can take B to be the Lebesgue sigma algebra and mu to be the Lebesgue measure. So for all n greater than or equal to 1, we can take the indicative function for the interval n, n + 1.

So when you take the limit, so first of all this is not an increasing sequence and when you take the limit, as n goes to infinity of fn, you get the 0 function, because pointwise this function fn we can say that fn escape to infinity horizontally. So this is a horizontal escape to infinity and there are other escapes to infinity that we will see later. So in this case, the pointwise limit is 0 and if you take the integral of fn, well it is integral over R of dm fn dm.

So this is nothing but the measure of the interval that we have chosen, which is n, n + 1 and this is nothing but 1. So the limit is also equal to 1. On the other hand, if you take the integral of the limit of fn dm over R, then this is nothing but 0, because this is the 0 function. So we see that the monotone convergence theorem fails when we have this escape to infinity problems and it does not satisfy the hypothesis of the monotone convergence theorem.

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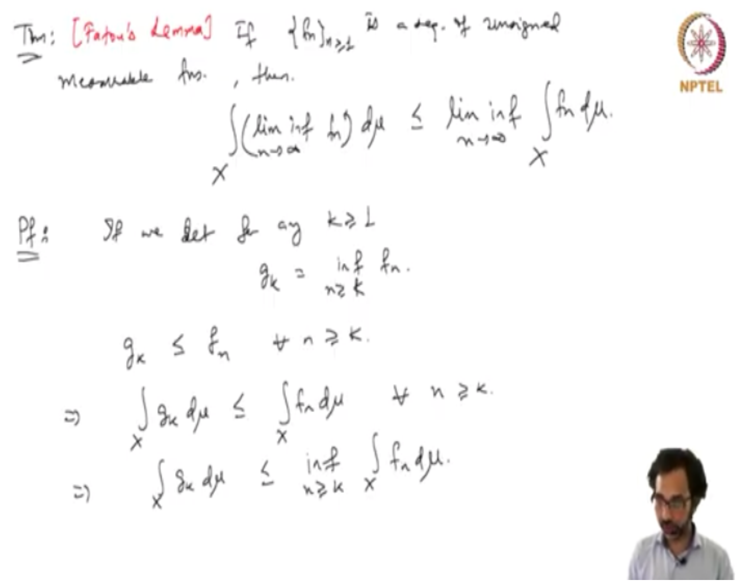

Nevertheless, we still have a result, which is called Fatou's lemma, but I will write it as a theorem. This is called Fatou's lemma, which is that if fn, n greater than or equal to 1 is a sequence of unsigned measurable functions. We do not need to assume that it converges anywhere. Just we have a sequence of unsigned measurable function, then the integral over x of the lim inf as n tends to infinity of this fn d mu is less than or equal to the lim inf as n tends to infinity of the individual terms integral fn d mu.

Now let us see the proof of Fatou's lemma, for which we will again use the monotone convergence theorem. So if we let for any k greater than or equal to 1, let us say gk to be the infimum of all the fn for n greater than or equal to k. So of course gk is less than or equal to fn for all n greater than or equal to k by definition and so the integral of gk d mu is less than or equal to the integral of fn d mu for all n greater than or equal to k, which means that the integral of gk d mu is less than or equal to the infimum over the set n greater than or equal to k of these integrals fn d mu.

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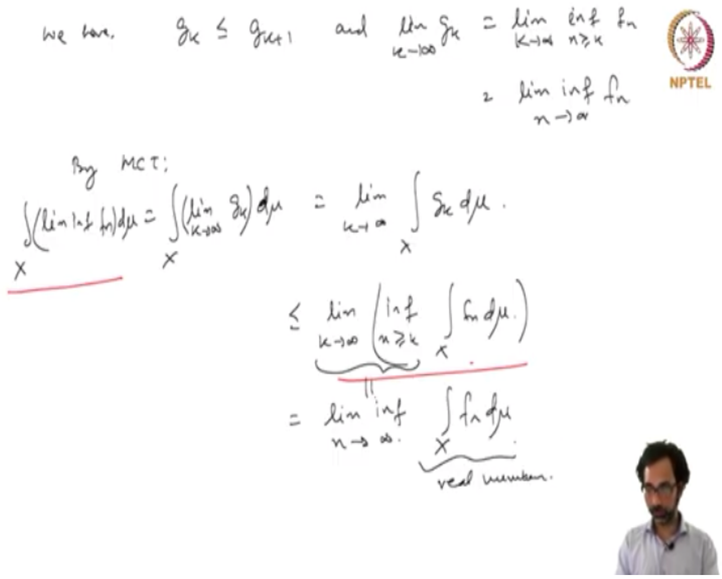

On the other hand, we have that gk is less than or equal to gk + 1, because we are taking an infimum over a larger set for gk and so gk is less than or equal to gk + 1 and it increases, it converges and the limit of gk as k goes to infinity is equal to the limit as k goes to infinity, infimum of n greater than or equal to k fn and this is nothing but the lim inf by definition of this fn.

So by the monotone convergence theorem, we have the limit as k tends to infinity of the gk, the integrals is equal to the limit as k tends to infinity of the individual gk and this is nothing but on the left hand side we have lim inf fn d mu and on the right hand side, we have the limit as k tends to infinity gk d mu, but this is less than or equal to the limit as k tends to infinity, the infimum of n greater than or equal to k, fn d mu. This is what we have proved right here.

So we have used this inequality here by monotonicity, we have that the limit for the first term and for the left hand inequality is less than or equal to the limit for the infimum over n greater than or equal to k of the integrals fn and this is again nothing but the lim inf of the numbers. So these are real numbers and we have by definition of the lim inf, this is nothing but this k goes to infinity and then infimum n greater than equal to k.

So we have established Fatou's lemma which is the integral of the lim inf of fn is less than or equal to the lim inf of the integrals of fn and this holds whether or not we assume that fn is a monotone sequence or not.