Measure Theory

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Lecture – 43
Lebesgue Integral of Unsigned Measurable Functions: Motivation

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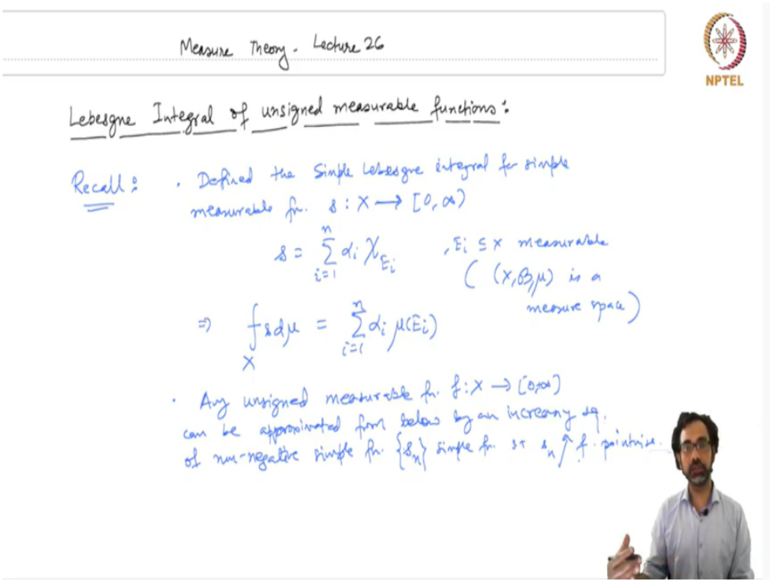

Now we come to one of the most important concepts in measure theory which is that of Lebesgue integral of an unsigned measurable function. So, as I said before Lebesgue developed the theory of measures and integrals so that one could have an advantage over Riemann integration and which could be applied to arbitrary sets as well. So, in this lecture we will look at the definition of the Lebesgue integral for unsigned measurable functions and then we will generalize this definition first to real value functions and then to complex value measurable functions.

So let us look at the definition so first recall that we have defined define the simple Lebesgue integral for simple functions simple measurable functions let us say s from a set so we will take a non-negative simple measurable function then it is of the form it is a finite sum of scalar values alpha i multiplied by the indicator functions of E is which are measurable subsets of this measurable space x.

So, then the simple Lebesgue integral s d mu is nothing but the sum i = 1 to n alpha i mu of E i so here x v mu is a measure space and so we have defined the simple Lebesgue integral.

Now we also have that any non-negative or rather unsigned we have defined what is an unsigned measurable function f from x to 0 infinity with plus infinity included can be approximated from below by an increasing sequence of non-negative simple functions.

So, we have sn a sequence of simple functions says that sn increases to f point wise which means that it is a non-decreasing sequence of simple functions and the limit point wise limit of this sequence is precisely f. So, we know we have defined the Lebesgue integral for simple functions and we know that there always exists a sequence of simple functions that convergence from below to any measurable unsigned measurable function f.

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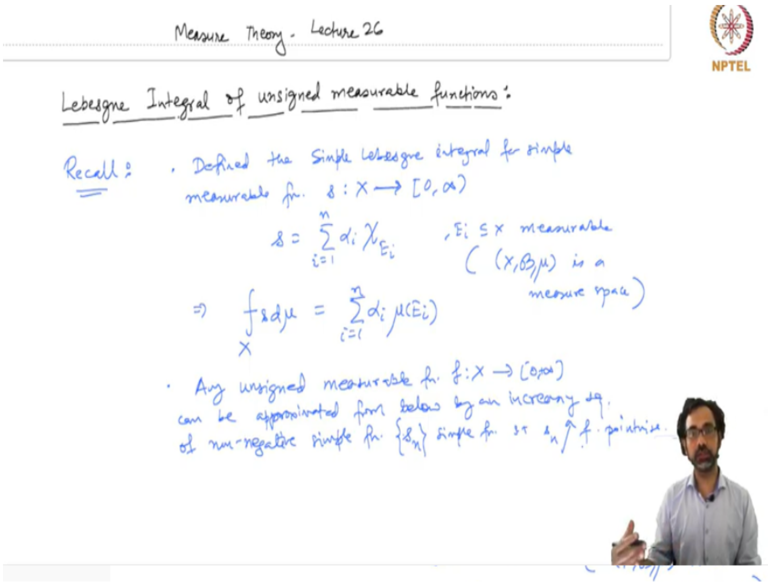

So, it makes sense to define the Lebesgue integral of an unsigned simple function so let so x B mu is a measure space and f be an unsigned measurable function then we define the Lebesgue integral x f d mu as follows. So, this integral of f d mu is by definition the supremum of simple functions which are point wise bounded above by the measurable function f and you take the simple integral of s.

So, it is the supremum of the simple integrals of simple functions f s which are bounded above point wise by f. So, note that this is the equivalent or rather the generalization of the lower Darboux integral where we took for a bounded function on a compact interval in r we took the piecewise constant functions which were bounded above by f and we took the piecewise constant Riemann Darboux integral of those functions.

So, this is a generalization of the lower Riemann Darboux integral. So now the question arises why we take the lower one and we do not define the first the lower and upper and when they two when the those two agree then we say that the Lebesgue integral exists as we did for Riemann integration. So, we have to say some words about the justification for using the lower integral as our definition for Lebesgue integral using the lower integral. So first is that if f is an unbounded function then there are no simple functions s such that f is bounded above by s. So, we can simply view it for example on r.

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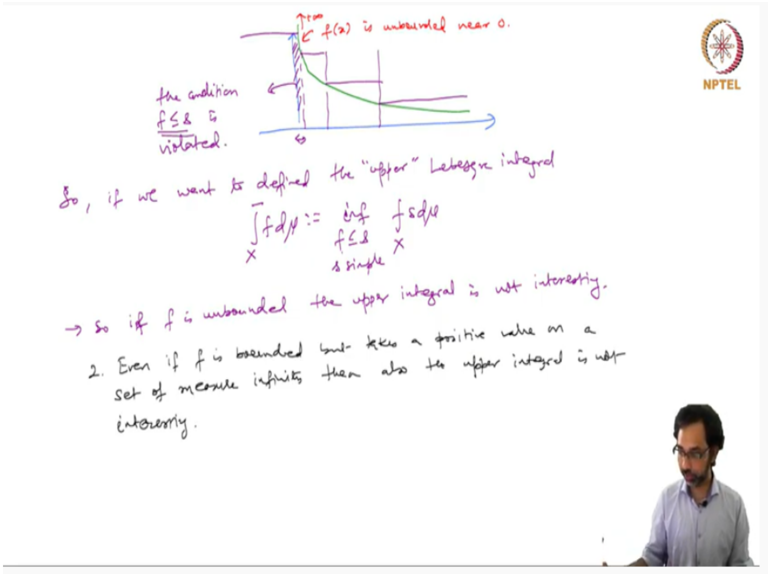

So, we have the real i let us say that we define our function only on the positive part and say that f is bounded unbounded near 0. So, this is our fx this is unbounded near zero so it goes to plus infinity near 0. So, if this is the case then any simple function you take which you want to be higher than f pointwise. So, suppose that we take here this value so remember that we have to take simple functions which are defined over all of r and then it has to be pointwise bounded above everywhere it should bound f everywhere point wise.

So, if you take for example this value here and then after some partition you take this value here and this value here and so on. But since we are only allowed finitely many we have to stop somewhere and for this reason the condition f <= s is violated. So we can try to define it for finitely many pieces of the real line measurable pieces but since we are only allowed to have finitely many at some point we will run out of space and some part of the unbounded part of f will remain uncovered or unbounded by this simple function s which we need to be higher than f for every value of x.

So, in this region we cannot have that f is bounded above by s so there are no simple functions even in the case of the real line and f unbounded near 0. We do not have any simple functions which are which bound above this point wise this function f. So, we have that unbounded functions if we wanted to define so if we wanted to define the upper Lebesgue integral which we can define for example as the infimum of f <= s as simple of the simple integrals of these simple functions s then this infimum then this is an empty set and you will get simply 0.

So, it is does not give you anything interesting on the other hand so in this case we see that the upper Lebesgue integral does not make sense. So, in this case so if f is unbounded so let me write it here. So, if f is unbounded the upper integral is not interesting in the sense that it only gives you value o. So, that is one reason why we take only the lower Lebesgue integral to define the our Lebesgue integral for unsigned measurable functions.

The other reason so this was our first reason that if f is unbounded then there are no simple functions f < = s. In the second reason we can also say that even if f is bounded even if f is bounded but takes a positive value on a set of measure infinity then also the upper integral is not interesting. So again, we can see an example.

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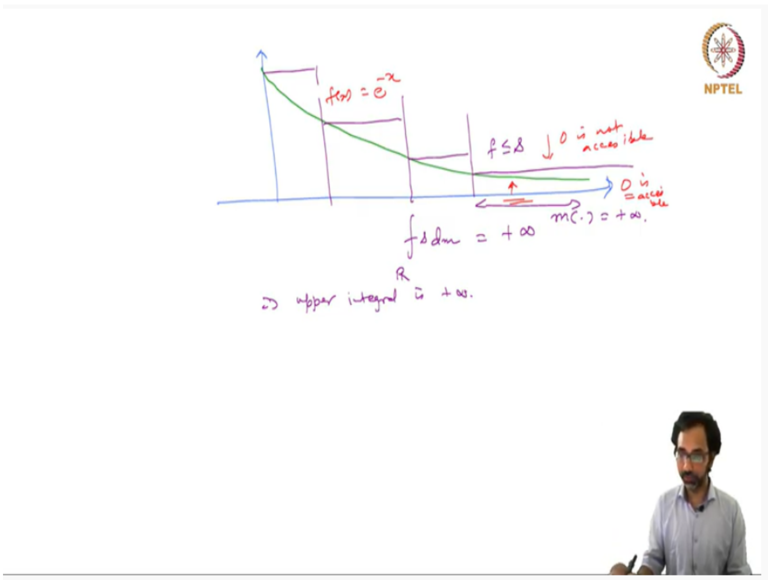

So now we allow bounded functions but it has to take a strictly positive value on a set of infinite measures. So, for example we can take this function decreasing to 0 for example we can take fx = e to the power minus x we know that this can be integrated over the positive real line but if you want to define it via the upper Lebesgue integral then again because we have only finitely many choices for the simple functions.

So, we can define it like this here like this here like this here and so on but at some point, you will have you have we have to cover an infinite portion of the real line with a finite value. So then if you write this function as s we will have s is bounded below by f everywhere but we have that the simple integral of s over r for the Lebesgue measure here is going to be plus infinity because this measure of this part is plus infinity.

So even if you have bounded functions but which takes a strictly positive value over a set of infinite measure then we see that the upper integral is plus infinity. So again it is not interesting so we see that the lower one makes sense always even if it is even if your function is unbounded or if it is bounded but with strictly positive values over a set of infinite measure because the of the fact that here if you want to take a lower integral then we have always that this 0 value is allowed for when you start to approximate f from below so because our 0 value is accessible here this function was strictly positive.

So, when you approximate from above 0 is not accessible but when you approximate from below 0 is accessible. So, when you have a set of infinite measure then you can just put a 0 for the simple function and then your lower Lebesgue integral will still give you finite values. So, this is these are the couple of reasons why our definition for the Lebesgue integral is only the lower Lebesgue integral and not the upper Lebesgue integral.

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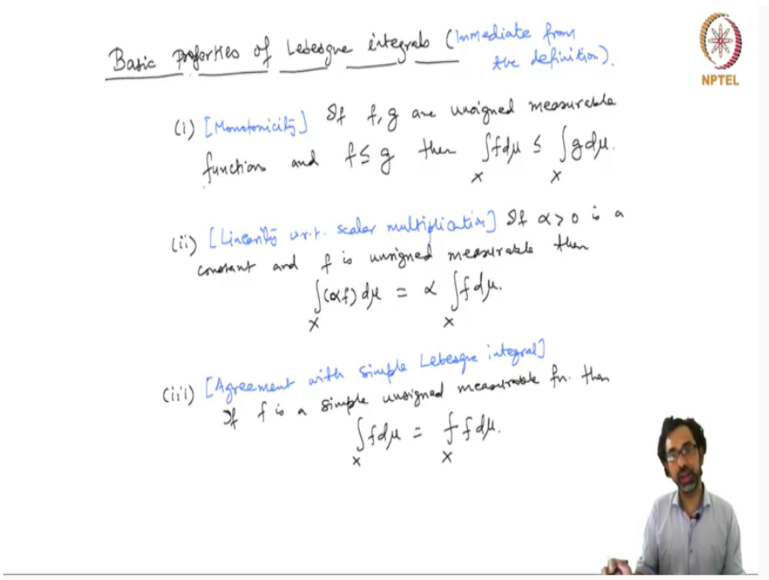

So now we look at some immediate basic properties for the Lebesgue integral which are immediate from the definition of how we have stated the definition for the Lebesgue integral. The first one is monotonicity which says that if f and g are unsigned measurable functions and f is pointwise bounded by g then the Lebesgue integral of f is less than or equal to the Lebesgue integral of g.

So, this is the monotonicity property the second one is linearity with respect to scalar multiplication. So, we have seen that the simple integral is linear with respect to scalar multiplication and of course this also holds for the Lebesgue integral for unsigned measurable functions. So, if alpha is a constant is a positive constant and f is an unsigned measurable function then the integral of alpha f d mu is equal to alpha times the integral of f d mu.

And the third is the agreement with the simple Lebesgue integral which says that if f is a simple unsigned measurable function then the Lebesgue integral as we have defined it right now is equal to the simple. So, the second one is the simple Lebesgue integral on the right-hand side and on the left-hand side we have the Lebesgue integral as we have defined using supremum of simple functions approaching from below.

So, these two agree so I will leave all three as an exercise. So, these are left as an exercise. So, you should just check from the very definition that we can deduce all three properties stated here.