Measure Theory

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Lecture – 42
Lebesgue Integral of Unsigned Simple Measurable Functions: Definition and Properties

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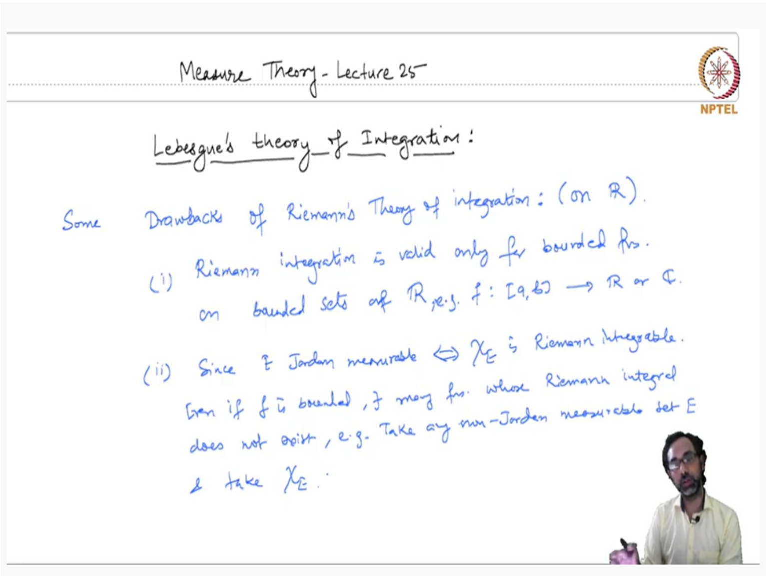

So now we have built up enough theory of Lebesgue measurable sets so that now we can come to the main and the most important aspect of Lebesgue's theory of Lebesgue's measure theory which is the theory of integration. So, as I mentioned before one of the main motivations for developing the theory of measures was because Riemann's theory of integration was not sufficient and it had some drawbacks and one of the main motivations for the Lebesgue's theory of integration and subsequently the Lebesgue's measure theory was to overcome these drawbacks in Lebesgue's and Riemann's theory of integration.

So, let me recall some drawbacks of Riemann's theory of integration some drawbacks of Riemann's theory of integration. So, the first point is that of course here I am only talking on for the Riemann's integrals on the real line R. So, first of all it is that a function so Riemann's theory Riemann's integration is valid only for bounded functions on compact sets or bounded sets of R.

So, if the function is unbounded then proper the theory of ordinary Riemann integrals is not sufficient and then one has to pass to what is called an improper Riemann integral but for ordinary Riemann integration only bounded functions are allowed and that to with support on bounded sets on R. So, there are of the form f from defined on finite interval a, b to R or c. So, this is one of the first drawbacks that it only allows for bounded functions defined on compact sets of R.

Secondly, we have seen that since E is Jordan measurable is equal into saying that chi E is Riemann integral. So, we prove this statement this is an if and only if condition. So, this means that even if f is bounded there exists many functions whose Riemann integral does not exist. Son for example take any non-Jordan measurable set E and take chi E. So, we have seen many examples of bounded sets which are not Jordan measurable.

So, the modified canter set was one example then the union of small enough intervals over the rationales this was another example of a boundary open set. So, these are not Jordan measurable therefore their integrative functions will not be Reimann integral. So, Riemann integration even if you take boundary functions then also the class of bounded functions for which it is Riemann integral integrable is not large enough.

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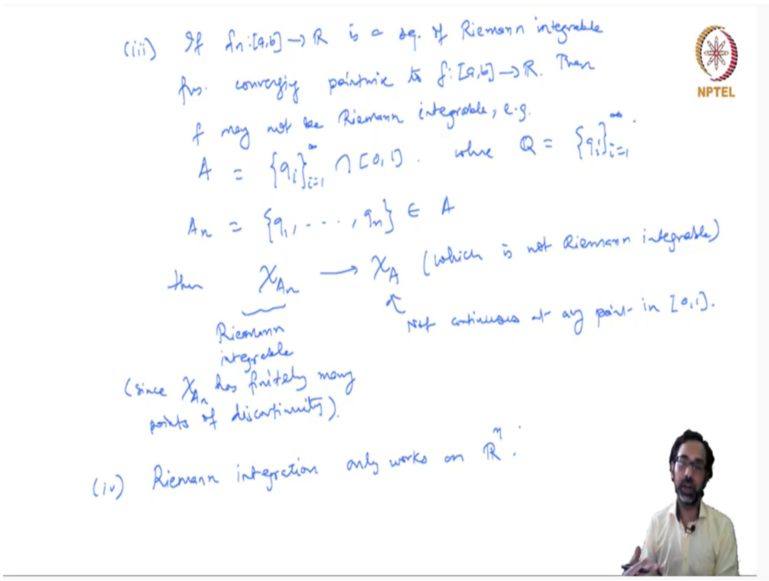

If fn say R to R is a sequence of Reimann integrable functions, then the point was limit so let us suppose that converging to converging point-wise. So here let me put a, b to R so we have fixed a finite interval on which all these references are defined so converging point-wise to f then f may not be Reimann integral. So, for example if you take an enumeration of the rationals qi i from 1 to infinity and inside let us say 0, 1 and take An to be the union of to be the set of the first n rationals in.

So let me denote this as A where q = this enumeration of all the rational numbers so you are only considering the rational numbers inside 0, 1. So if you take An to be the rationales inside A inside 0,1 but only finitely many then chi n converges to chi A and this is so each of these are is Riemann integrable because Riemann integrable because it has only finitely many points of discontinuity.

Since chi An has finitely many points of discontinuity. On the other hand, if you take, there indicate the function for the all the rationales in 0, 1 this is not integrable this is not Reimann integrable because it is a this is a no where continuous function it is not continuous at any point in 0,1 so we see that point-wise limit of bounded Riemann integral functions may converge to a function which is not Reimann integral.

So, this is another drawback for Reimann's theory and last but also not least is that Riemann's theory Riemann integration even if you consider improper Riemann integrals which do allow some unbounded functions Riemann integration only works on R or let us say Rn so it only works on Euclidean Space Rn.

So, we see that when we have constructed an abstract theory of measures, we can also define integration on an abstract set abstract measurable space and then we will have a very nice theory of integration which more or less will address almost all these shortcomings that we have listed here.

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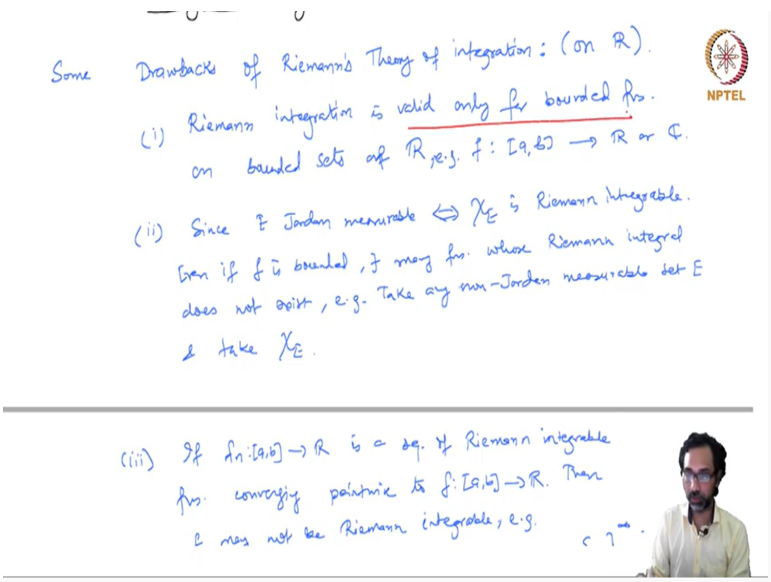

So first one is that it is valid only for bounded functions on compact sets second is that even if it is bounded it may not be Riemann integrable and third is that a sequence of Riemann integral functions may converge to a function which is not Riemann integral. So, and of course the last we have seen that it only restricts the theory to Rn and what we will propose the Lebesgue theory of integration which will work not only for any abstract measurable space.

But also, most of these issues that we face here will be addressed of course giving some additional constraints but nevertheless they allow for a much larger class of functions to be integrated.

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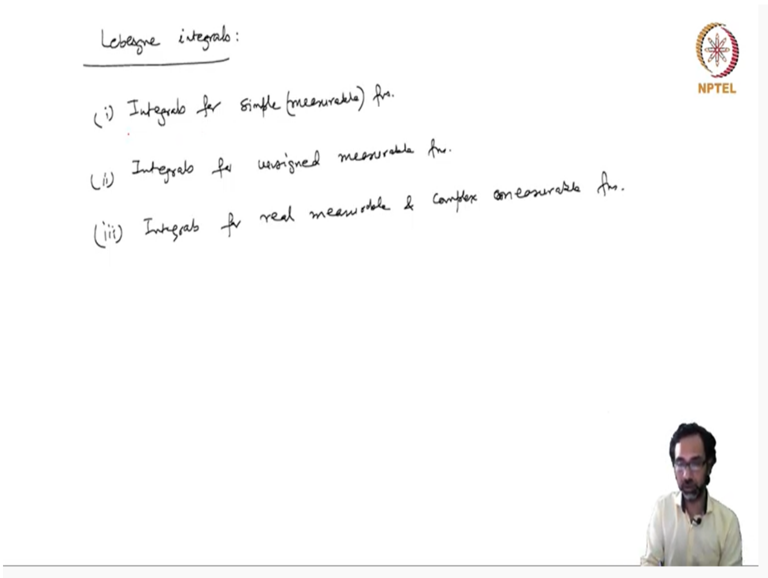

So, let us come to Lebesgue's theory of integration the Lebesgue integrals. So first we will define integration integrals for simple functions simple measurable functions. So, whenever I say simple functions, I will assume that it is measurable and then we will define integrals for unsigned measurable functions and then we define integrals for real measurable and subsequently complex measurable functions.

So, we will follow this step so this is first this is second and this is the third step in which we will step by step define the notion of a Lebesgue integral. So, let us start with integrals for simple functions.

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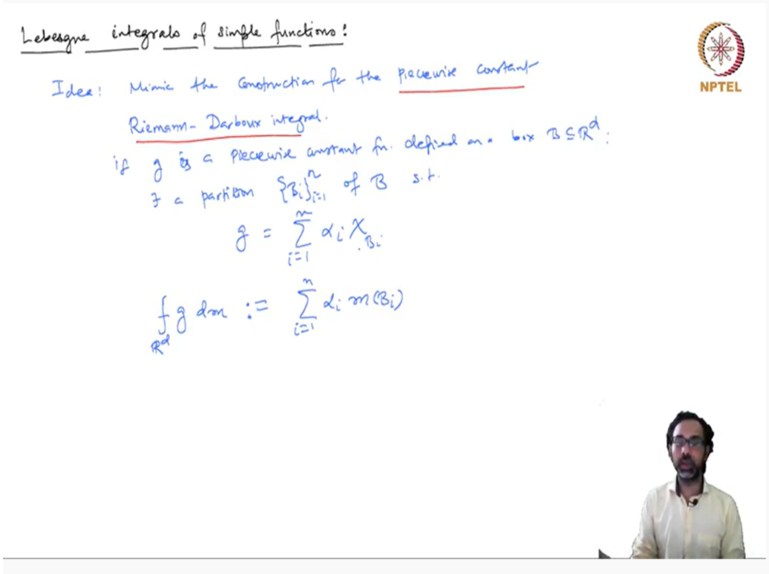

So, the idea here will be to mimic the construction for the piece wise constant Riemann Darboux integral idea is to mimic the construction for the piece-wise constant Riemann Darboux integral which takes piece-wise constant function g. So, if g is a piecewise constant function defined on a box B in Rd then there exists a partition Bi i = 1 to n finite partition of B into boxes such that g can be written as alpha i chi of Bi I = 1 to n.

So, this was a piecewise constant function and the piece-wise constant Riemann Darboux integral on Rdg dm is by definition then i = 1 to n alpha i the measure of Bi. So, this was the definition of that piecewise constant Riemann Darboux integral which is defined like this. So, we will now allow first of all we will take a simple function which will be of the form alpha I chi of ai Ai's may not be boxes but they will be definitely Lebesgue measurable functions and then we can use a similar formula to define what is called the Lebesgue integral.

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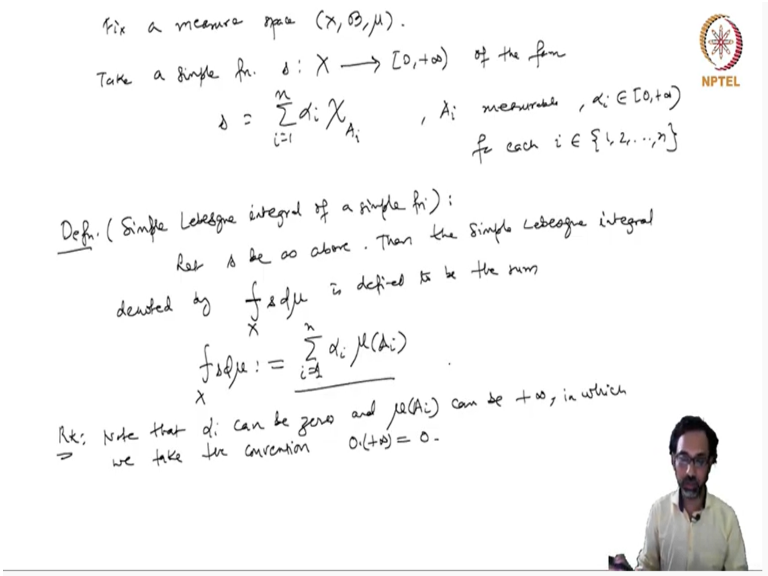

So, this is the idea for simply how to we make the piece-wise constant Riemann Darboux integral. So, first let me fix i will work on fix measure space x, B, mu and take now a simple function an unsigned simple function. So, s is a map from x to 0 + infinity but not including + infinity of the form s = sum i = 1 to n so it is a finite sum of a finite linear combination of indicative functions Ai measurable and alpha i belongs to the positive non negative real numbers for each I 1, 2 up to n.

And so, this is it simple function so then we can define the simple Lebesgue integral of a simple function. So, let s be as above then the simple Lebesgue integral denoted by a integral sign over x but with a cross sign s d mu is defined to be the sum i = 1 to n alpha i mu of Ai and this is by definition this simple Lebesgue integral for the simple function s. So, here note that alpha i can be 0 and mu of Ai can be plus infinity because we are no longer working on boxes.

So the measures can be can be plus infinity and it can be 0 s can take the value 0 on a measurable set of measure plus infinity in which case we take the convention 0 times plus infinity is 0 so that when the simple function takes the value 0 on a measurable set of infinite measure it does not contribute to this some alpha i mu Ai so it will be 0. So, this is the definition for the simple function.

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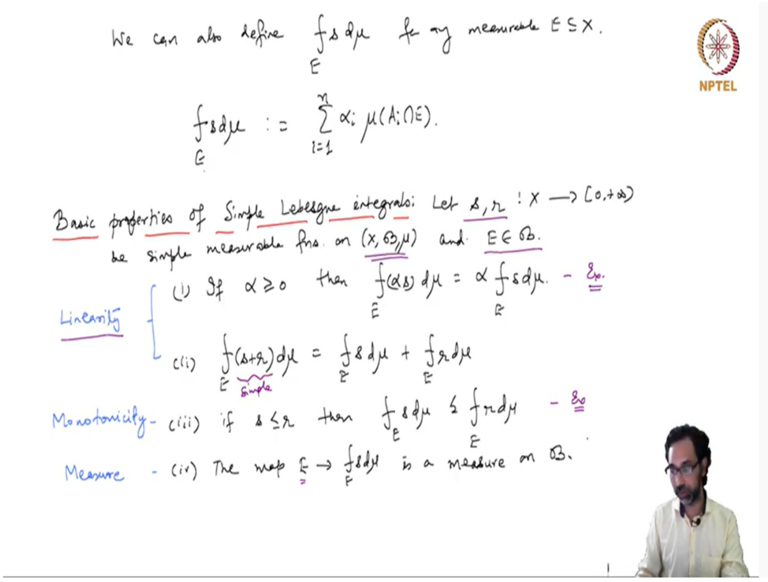

We can also define this integral s E d mu for any measurable subset of x E of x and this is defined as so in the first instance we defined it for the whole space x but it is simply a very easy modification to define it for over a any measurable subset E of x and here you take the sum alpha i mu of Ai intersection E again with the same condition convention that 0 times plus infinity is 0.

So now let us look at some basic properties for the simple Lebesgue integral. So, we take 2 simple functions s and R they are simply measuring functions on some measure space x d mu and we fix a measurable subset E of x then the first 2 properties is the linearity property. So, if alpha is a scalar and non-negative scalar then the function the integral of the simple function alpha s equal to alpha times the simple integral of s.

So, this is very easy and I leave it as an exercise for you to do. The second one is that the simple integral of the sum of 2 simple functions which is again a simple function so this is a simple function is equal to the sum of the simple function simple integrals of s and R. So, this is these 2 the first 2 properties taken together give you the linearity property for the simple Lebesgue integral.

The third one is monotonicity which says that if s is point-wise bounded by R then the simple Lebesgue integral of s is bounded by simple Lebesgue integral of R. So this is also left as an exercise because it is quite easy and lastly the measure where the map this is the measure property which the map which assigns to any measurable set E there is simple Lebesgue integral s for a fixed central function s this is this defines a measure on B which means that it is accountably edited measure on B. So also, I will leave this the last property as an exercise so I will just prove this second property here.

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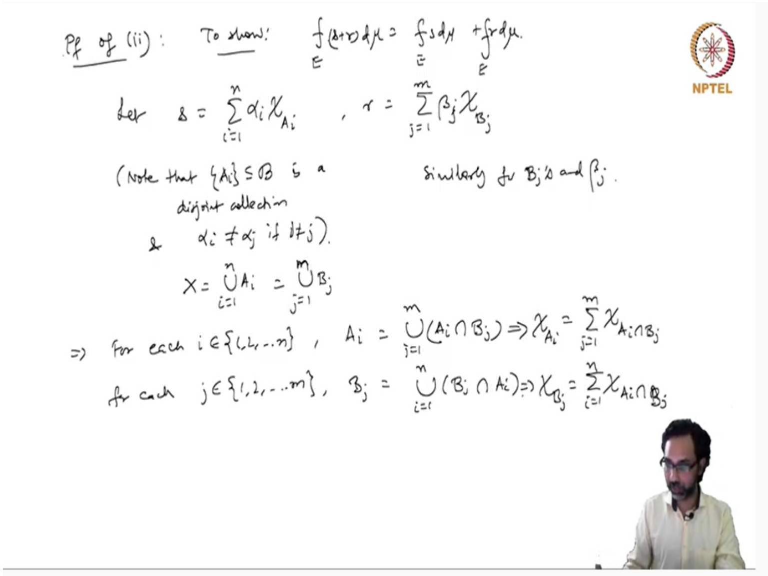

And let us look at that proof for 2 which is that if so, this is what we have to show that the simple integral of x + R d mu = s d mu plus simple integral of R d mu. So, let us suppose that s is written as alpha i chi of Ai i = 1 to n and R is a simple function j 1 to m different maybe different from this n for defined for s and we can have beta I chi of Bj. So, beta j chi of Bj. So these are the representations of simple functions with respect to measurable functions Ai's and Bj's scalars non-negative scalars alpha i's and beta j's note that this collection Ai is designed collection and alpha i is not equal to alpha j if i is not equal to j.

So these are distinct values and similarly for beta Bj's and beta j. So, we have that this collection of disjoined unions of Ai's and Bj's is given by the whole space x. So, x can be written as the union of Ai's and also the union of Bj's which means that for each fixed i for each i in 1, 2 up to n Ai is the union j = 1 to n Ai intersection Bj. So, we can decompose each Ai in this way and similarly for each j fixed j in 1, 2 up to n we have Bj is the union of i = 1 to n Bj intersection Ai.

So, the first one for example implies that chi of Ai is the sum j = 1 to n chi of Ai intersection Bj and the second one implies that chi of Bj is similarly i = 1 to n chi of Ai intersection Bj. So, let us see what is the integral of s + r and what is the RHS outages which is the sum of integrals of s and R.

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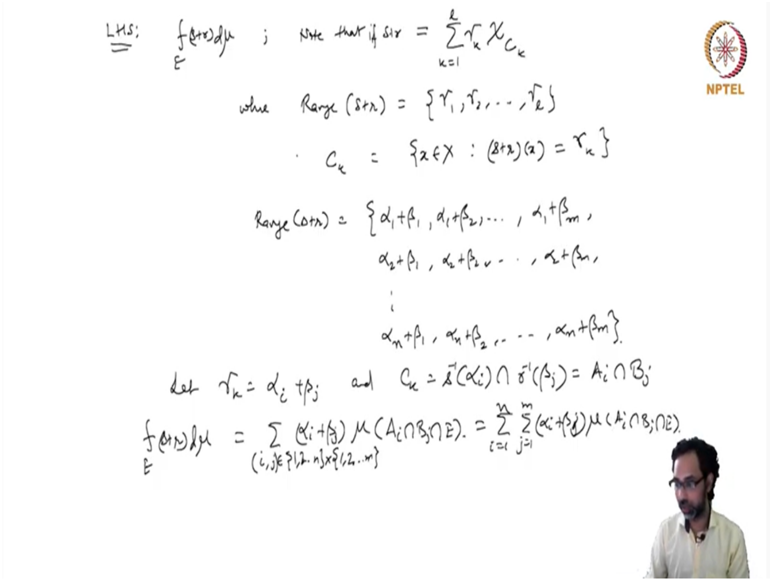

Now let us try to compute the LHS so this is the simple Lebesgue integral or s + r and note that s + r is also a simple function and what we need is a representation of so let me say if s + R is represented as a sum k = 1 to l gamma k chi of c k where the range of s + r is gamma 1 gamma 2 and so on up to gamma l and ck is equal to the set x set of points in x such that s + Rx = gamma k.

So now since we know the ranges of s and r individually the set of gamma case is easy to compute which is simply the set alpha 1 + beta 1 + then alpha 1 + beta 2 + alpha 1 sorry is that commas alpha 1 + beta m and then alpha 2 + beta 1 alpha 2 + beta 2 and so on alpha 2 + beta m and so on then we have alpha n + beta 1 alpha n + beta 2 and so on to alpha n + beta n. So, these are the possible ranges for the for the function s + r and so let gamma k = alpha i + beta j.

So of course, one has to reorder this set up to 1 to l and then we have to write gamma k for alpha i + beta j. So, gamma k + alpha i + beta j so and ck is exactly the same as set s inverse alpha i + s inverse at alpha I intersection r inverse beta j which is nothing but Ai intersection Bj. So, the integral of s + r d mu is going to be equal to the double sum I j. So i j belonging to the set 1, 2 up to n cross 1, 2 up to m then you have alpha i + beta j and then you have the measure new of Ai intersection Bj intersection E and this is nothing but the double some i from 1 to n j from 1 to n alpha i + beta j and then you have the measure Ai intersection Bj intersection E. So, this is the RHS the LHS which is the LHS.

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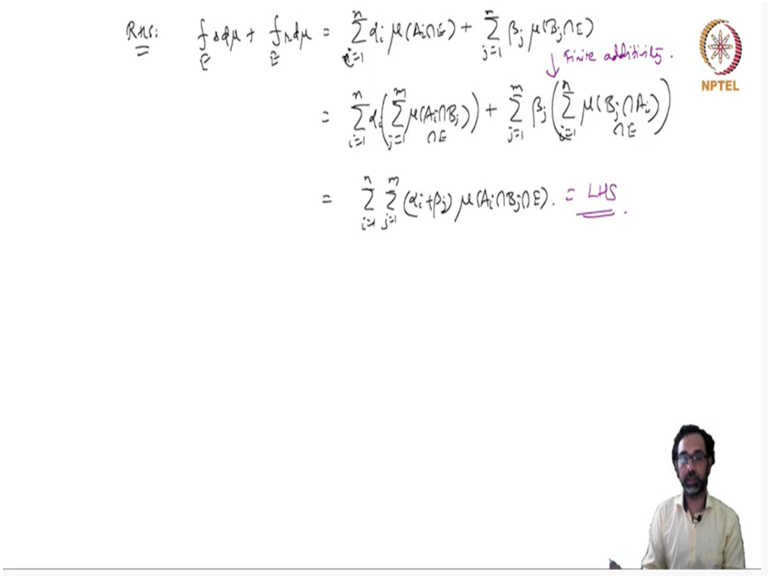

And now for the RHS we have individually s d mu + r d mu which is nothing but the sum i = 1 to n alpha i mu Ai + j = 1 to m beta j mu Bj and now we are going to use the finite relativity property alpha i mu Ai can be written as j = 1 to m mu Ai intersection Bj + j = 1 to m beta j and then again you can write i = 1 to n mu of Bj intersection Ai. So, now we can collect the terms so both are intersected with E here also here.

So, everywhere you have intersection further intersection with E and so we can collect the terms with Ai and Bj so you will get again the double sum 1 to m alpha i + beta j and then you have mu of Ai intersection Bj intersection E. So, from the first line to the second line we used finite additivity of the measure mu and so we see that this is also equal to the LHS and so we are done. So, this establishes the first few basic properties of the simple Lebesgue integral.