Measure Theory
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Lecture - 41
Egorov's Theorem: Abstract Version

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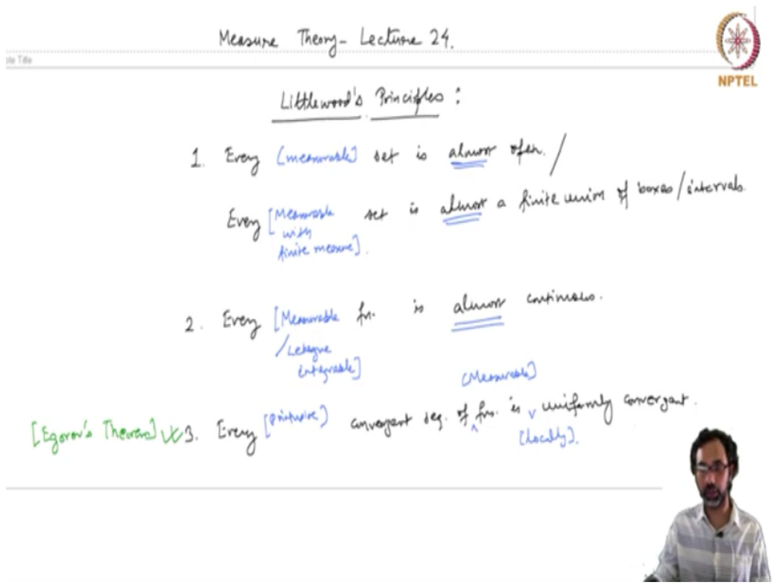

So now that we have seen the definition of measurable functions and we have also seen some basic properties of measurable functions. So it is now a good time to have some intuitive idea of what a measurable function should be and what point wise convergence of measurable functions should be. So this is encapsulated in what are called Littlewood's principle. We have already seen one Littlewood principle when we were defining Lebesgue measurability, which stated that every measurable set.

So I am putting measurable in brackets, because the idea is to write that every set is almost open and this is what we used to define Lebesgue measurability and in fact, we can also say something stronger, which is that every set is almost a finite union of boxes or in the one dimensional case, intervals. So here again there is some cache, which is that it should be measurable set with finite measure.

So it is actually nice exercise to show that every measurable set with finite measure can be approximated with respect to the symmetric difference with a finite union of boxes, such that the measure of the symmetric difference of the set with this finite union of boxes is less than or equal to some given epsilon. In this way, Littlewood's first principle gives you an intuitive idea of what a measurable set should look like and what a measurable set with finite measure should look like.

So then Littlewood's second principle says that every function is almost continuous and here also the cache is that it should be a measurable function or we will see that it also works if it is a Lebesgue integrable function. So once we define, what is a Lebesgue integrable function? We will see that these functions are almost continuous. So of course, we make precise what is meant by this almost, but we will see that it is up to some epsilon or up to some finite tolerance in the measures.

We have to state these properties for being almost continuous and almost open. So the second principle says that every measurable function or a Lebesgue integrable function is almost continuous. So you can view a measurable function as a continuous function outside a set of negligible measures. On the other hand, the third principle says that every convergence sequence of functions is uniformly convergent.

So again there is a cache, which is that it should be a pointwise convergence sequence of measurable functions and here in general one can only have uniform convergence locally, which means that it will be uniformly convergent once you take the intersection with a bounded set, bounded Euclidean ball. So these three Littlewood's principles, these are some idea of how one should think about pointwise convergence of measurable functions.

How one should think about absolutely integrable or measurable functions and how one should think about measurable sets. So in this lecture, we will see the third point, which is that every pointwise convergence sequence of measurable functions is uniformly convergent. So this is known as, this result when you make it precise, is known as Egorov's theorem and we will see what is the Egorov's theorem and we will try to prove it in this lecture.

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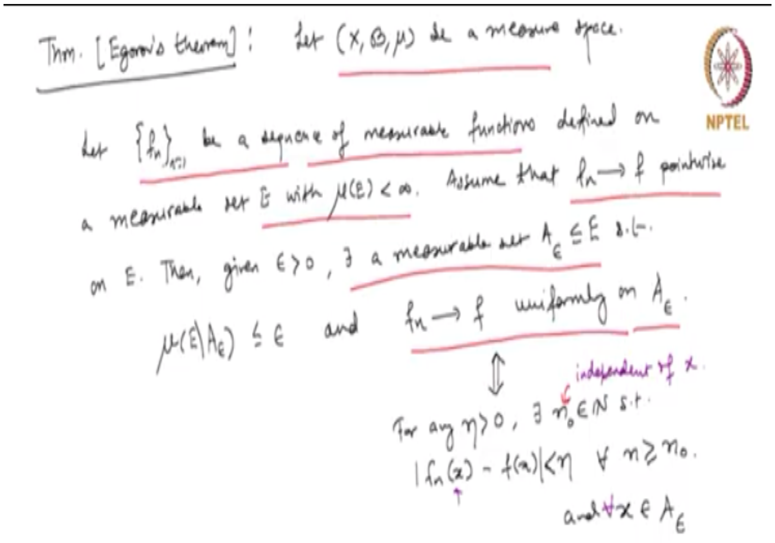

So let us see the statement of Egorov's theorem. So we start with measure space x, B, mu and we take a sequence of measurable functions fn defined on a measurable set E with finite measure mu E and assume that fn converges pointwise to some function f on E. Then, Egorov's theorem states that given epsilon greater than 0, there exist a measurable function A epsilon, which is a measurable subset of E, such that the measure.

So here should be mu, measure of E – A epsilon is less than or equal to epsilon, which means that the compliment of A epsilon inside E has very small measure and fn converges to f uniformly on E. So recall that fn converges to f uniformly on A epsilon means that for any epsilon greater than 0. So this epsilon is different from the given epsilon. So let me take something else, eta greater than 0.

There exists a natural number n, such that let me write n naught, such that modulus of fn x – fx is less than eta for all n greater than or equal to n naught and x in A epsilon. So the point is that here the difference between uniform convergence and pointwise convergence is that we can choose this n naught, this is independent of the chosen point x. So this holds for all x in this set A epsilon.

So we see that outside a set of measure less than or equal to epsilon, we have uniform convergence and this is what Littlewood's third principle says that, every pointwise convergent sequence of measurable functions is uniformly convergent. So of course, we have to make sure that our assumptions are correct, which is that this convergence should be on a measurable set with finite measure and then we can make the statement that it is uniformly convergent outside set of measure less than or equal to epsilon. So let us see the proof of Egorov's theorem.

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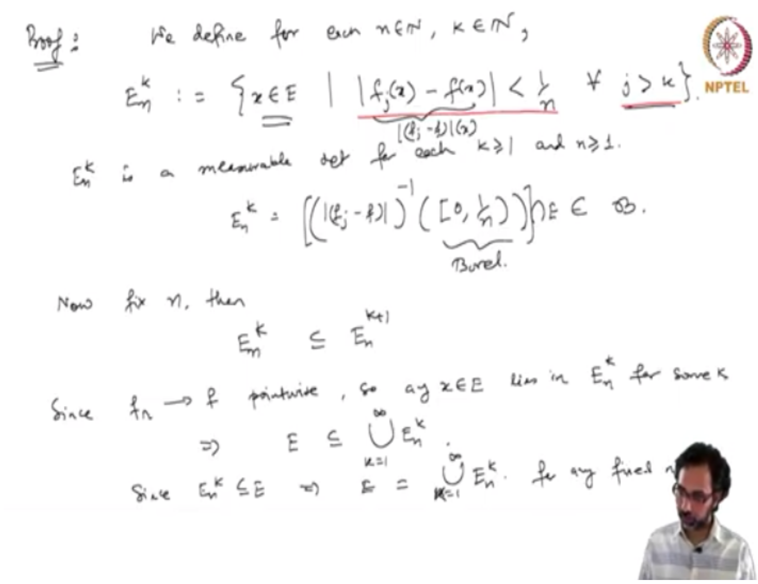

Proof, so we define for each n, a natural number and k a natural number, E n k, which is the set of all points in E, such that the modulus of f jx – fx is less than 1 over n for all j greater than k. So we are looking at those points, for which f jx becomes close to fx, for all j greater than k up to a tolerance of 1 by n. So this E n k is a measurable set for each k greater than 1 and n greater than 1, because first of all this is f j – f, f is a measurable function, because f j are all measurable.

So this f j – f is a measurable function. On the other hand, it is a composition with a continuous function, which is a modulus. Therefore, the modulus of f j – f is also continuous, is also measurable. So we are taking the inverse image, so E n k is the inverse image of the function f j – f mod of the interval 0 to 1 by n, which is open at 1 by n and of course, this is a Borel set and so this is a measurable set.

Of course, one should take the intersection with E, because the E n k should be a subset of E. Nevertheless, E is measurable, so E n k are all measurable. Now we fix n, then we have that E n k is a subset of E n k + 1, because if this inequality holds for j greater than k, then it also holds for j greater than k + 1 and so we have this inclusion of E and k inside E and k + 1. On the other hand, since fn converges f pointwise any x in E lies in E n k for some k.

Fixing n, we can always find a k large enough such that x belongs to E n k, because fn converges pointwise to f, so this difference f jx – fx modulus is going to be less than 1 by n for a large enough k, so this implies that E is a subset of the union of all the E n k, k = 1 to infinity and since E n k is a subset of E by definition, we are only taking points in E, which implies that E is actually equal to the union of all these sets E n k. So this is for any fixed n greater than or equal to 1.

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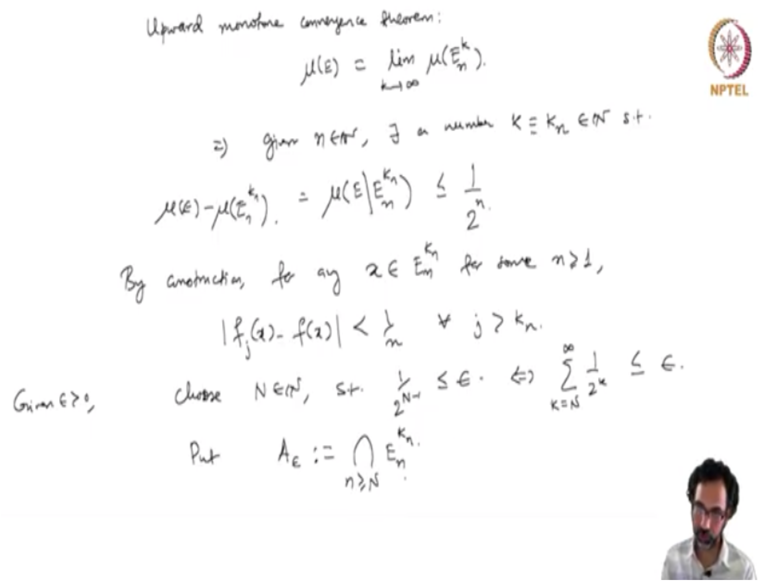

So now I am going to use the upward monotone convergence theorem, which states that the measure of E is equal to the limit as k tends to infinity of the sets E n k, sorry mu E n k. This is the statement for the upward monotone convergence theorem. This implies that given epsilon greater than 0, there exists a number k, which I will write as k n belonging to the natural numbers given n greater than 0, n a natural number.

There exists a number k n such that we have that the measure of the set E – E n k n is less than or equal to 1 over 2 to the power n, because of this limit condition, because of this upward monotone convergence theorem we have this and this is nothing but mu E – mu E n k n. So by construction, we have that for any x belonging to E n k n for some n greater than or equal to 1. We have that the modulus of f jx – fx is less than 1 over n for all j greater than kn.

This is how we define the sets E n k and so this holds by our construction of the sets E n k. Now choose N in the natural numbers such that 1 over 2 to the power N – 1 is less than or equal to epsilon. So given epsilon greater than 0, we choose a natural number N, such that 1 over 2 to the power N – 1 is less than or equal to epsilon and note that 1 over 2 to the N is nothing but the sum from N to infinity 1 over 2 to the power k and this is less than or equal to epsilon.

So we put now A epsilon is now defined as the intersection of the sets n greater than or equal to N of E n k n, which means that any point in A epsilon lies in all of these E n k n for n greater than or equal to N.

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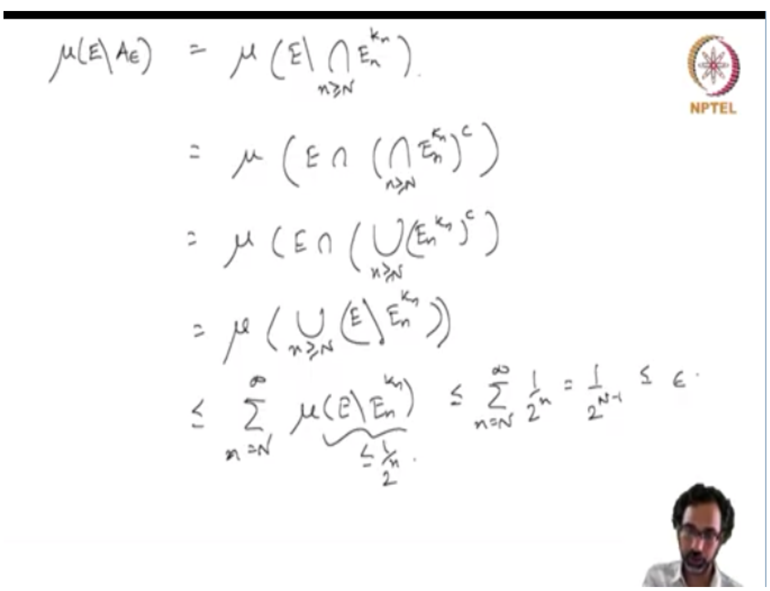

So then we can estimate the measure of E – A epsilon, which is equal to the measure mu of E minus the intersection of n greater than or equal to N E n k n and this is equal to the measure of E intersection n greater than or equal to N E n k n compliment and this is equal to the measure of E intersection the union n greater than or equal to N E n k n compliment, which can be written as the union n greater than or equal to N E – E n k n and this can be made smaller than the sum n = N to infinity.

So these are from N to infinity, so N to infinity measure of E – E n k n and each of them was less than or equal to 1 over 2 to the power n, so this is less than or equal to the sum n = N to infinity 1 over 2 to the power n and this is precisely 1 over 2 to the power n – 1 and we have chosen this to be less than epsilon. So outside this set A epsilon, the compliment of A epsilon in E as measure less than or equal to epsilon.

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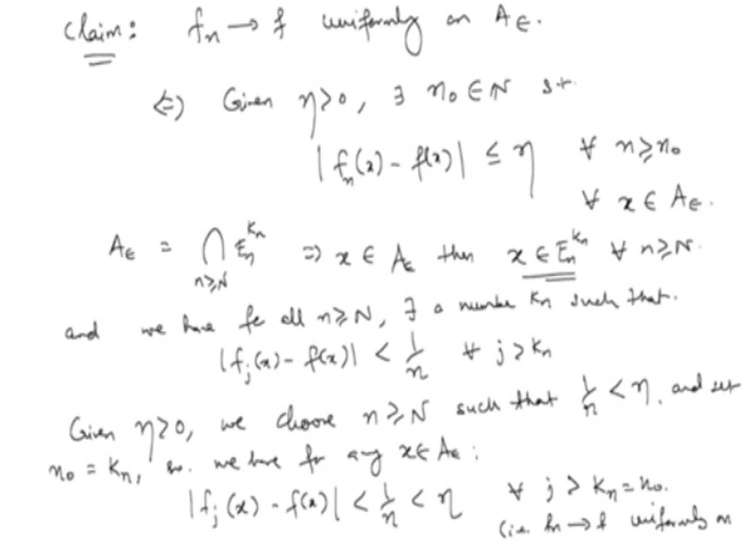

So now we claim that fn converges to f uniformly on A epsilon. So in other words, given eta greater than 0, there exists a number n naught in N, such that modulus of f j or f nx – fx is less than or equal to eta for all n greater than or equal to n naught and for all x in A epsilon. So this is the definition of uniform convergence on the set A epsilon. Now recall that A epsilon was in fact the intersection of these E n k n, which means that if x belongs to A epsilon, then x belongs to E n k n for all n greater than N.

And we have that f jx – fx modulus is less than 1 over n for all j greater than k n and this holds for all n. So this is for all n, greater than or equal to n there exist a number k n such that this holds, because x belongs to E n k n and now if given eta greater than 0, we choose n greater than or equal to N, such that 1 over n is less than this eta and set n naught to be this k n that we get from fixing an n and then getting kn.

So we have for any x in A epsilon that the modulus of f jx – fx is less than 1 over n, which is less than eta and this holds for all j greater than k n, which is n naught and this proves that fn converges to f uniformly on A epsilon. So we have shown that given pointwise convergence sequence on a set of measurable set of finite measure, then we can find a set A epsilon, on which it converges uniformly and the exceptional set for which it does not converge uniformly as measure as small as we want for a given tolerance level epsilon.

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On the other hand, we remark here that we cannot upgrade to the case mu E – A epsilon = 0 and such that fn converges to f uniformly on A epsilon, because we take this example functions defined on R and taking values on R and in fact we do not need it to be defined on the whole way line, we will only define it for 0 1 and it will also take values in 0 + infinity. So fn of x is defined to be 1 over nx for x belonging to the interval 0 1, which is open at 0 and closed at 1 and 0 for x = 0.

So we have defined a sequence of functions on a measurable set of finite measure, which is 0 1 and these are all non-negative unsigned measurable functions. So we can show that fn converges to the 0 function on 0 1. This is pointwise. This is pointwise convergence. Now suppose that there exists a set A, so here there is no dependence on epsilon, so you can remove it here. Suppose there exists such a set A on which fn converges to f uniformly and the measure of E – A = 0, but if you have uniform convergence, then there exists an n naught belonging to n, such that the modulus of f nx – 0, so you can forget about that.

So just the modulus of f nx is less than or equal to epsilon for all x in A. So any epsilon greater than 0, we can find an n naught independent of the point x, such that this holds for all n greater than or equal to n naught and for all x belonging to A. But the modulus of f nx cannot be small if x is taken to be 1 over n naught, then the modulus of fn naught x = 1 over n naught times 1 over n naught, which is simply 1 and so it violates this condition, we get a contradiction.

Therefore, the measure of the set E minus the set on which there is uniform convergence is atleast greater than or equal to 1 over n naught. Therefore, we cannot hope to have measure 0, the exception is said to be measure 0 outside of which there is uniform convergence.