Measure Theory
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Lecture - 40
Measurable Functions: Definition and Basic Properties - Part 2

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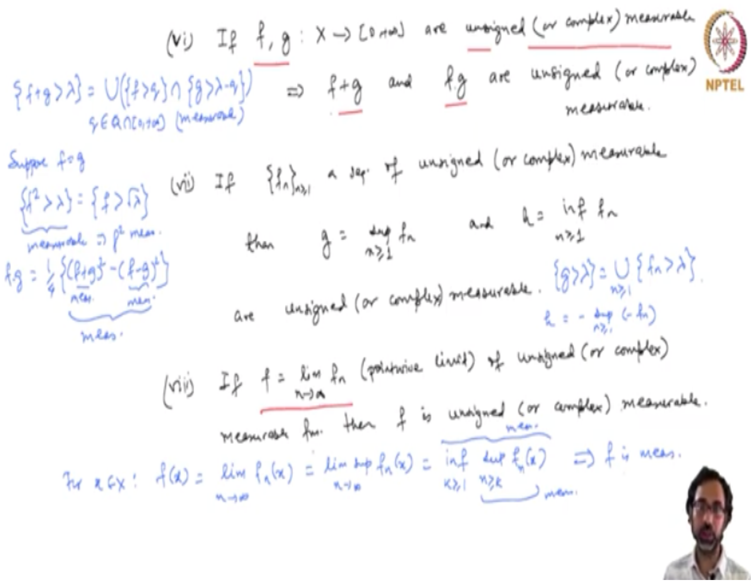

Now let us come to the next three properties. The first one says that if you have two unsigned or complex measurable functions f and g, so we have either unsigned measurable or complex measurable, then the sum and the pointwise term and product are unsigned measurable or complex measurable. So because complex measurability depends on real measurability, which further depends on unsigned measurability.

So it suffices to show their unsigned measurable case and the complex measurable case will follow and here we can simply write. For example, f + g and if you take the set f + g greater than some constant lambda, then this can be written as the union of the sets f greater than some number q intersection g, then should be greater than lambda – q and if you take q to be the rational inside the extended real, then you get a countable union and each one of them of these sets are measurable.

So these are all measurable sets because f and g are measurable and so therefore f + g is measurable. Now for the second part for the pointwise product, first let us suppose that f = g, so if you can prove that f square is measurable, then the pointwise product of f with itself is measurable, then we can deduce that fg is measurable. I will show you how. So first suppose that f = g and then we can write f square greater than lambda.

The set is equal to f greater than square root of lambda, because lambda is a non-negative real number, the square root is unambiguously defined. So this is measurable and so f square is measurable and now you can write fg as you can use the following formula f + g whole square minus f – g whole square and so first that the sum is measurable, then the square is measurable and the sum again this is the sum of two, now no longer unsigned.

But nevertheless, you can view it as a complex valued measurable function and then you can take the square and so these are all measurable and so the whole thing is measurable. Of course, the multiplication by a scalar does not affect measurability as well. So this is measurable. So this is the sixth part, this is 7 and this is 8 and now if you take the seventh one. For example, if you take the supremum, so you can take g greater than lambda as the union n >= 1 and then you can take fn greater than lambda.

So the supremum greater than lambda means that at least one fn must be greater than lambda and the reverse inclusion is obvious. So we have this equality when g is the supremum and similarly, if you take h, then h is the supremum of – fn and so again we are reduced to the case of supremum, no longer unsigned, but still complex measurable functions and so we have h is also measurable. So both supremum and infimum are measurable functions.

Now lastly, if you have a pointwise limit of measurable functions, each fn is unsigned measurable, then f itself is unsigned measurable. So let us see why this is true? This is again simply because fx is the limit as n goes to infinity of fnx for each x. We have this formula and limit is precisely well defined when you have equality of the lim sup and lim inf. So this is the lim sup in particular of fnx, which is infimum over k of the supremum of n >= k fnx.

So again this part is measurable and this part is also measurable, so the whole thing is measurable. So we see that we can deduce many properties from the basic properties of measurability.

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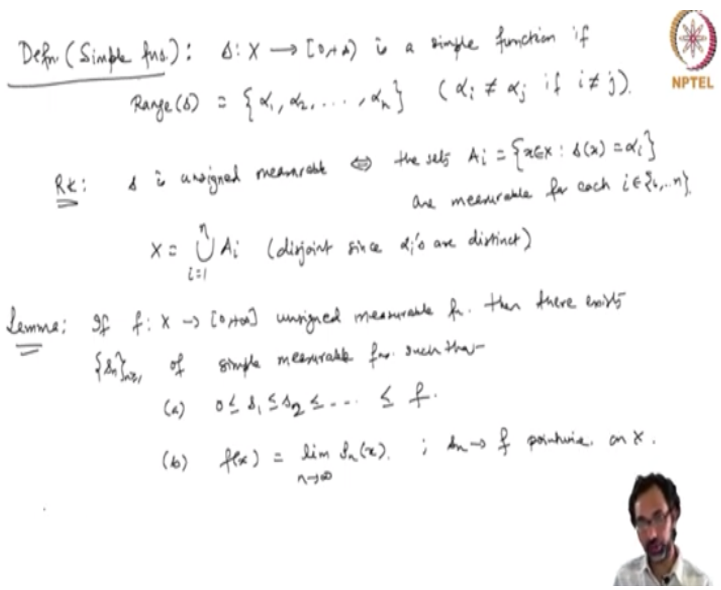

Now let me define what are simple functions? So function s from x to the non-negative extended real is called a simple function, if the range of x takes on finitely many distinct values, alpha 1, alpha 2 up to alpha n. So of course, we have the immediate remark that s is unsigned measurable, if and only if the sets Ai, which are the preimages of this values alpha i are measurable for each i in 1 to n.

If the function s takes a constant value alpha i on a measurable set Ai and your x can be written as, suppose here x can be written as the union of i going from 1 to n, Ai, then so fx can be decomposed into measurable sets Ai finitely many. In simple functions, we do not allow it to take the value plus infinity. So it is only defined for non-negative real numbers, but not the extended real numbers.

So we only allow the sets Ai, where sx is a finite real number from 0 to infinity, then these sets are measurable and x can be decomposed as finite union of this Ai. In fact, they are going to be disjoined because alpha i are distinct. So these are very much like the piece wise constant functions that we have dealt with when we were defining Riemann integrals, there we had boxes, the piece wise constant function take to constant value over a partition.

The partitioning was for boxes only, but here we are allowing partitions with measurable sets Ai and then it is more or less the same definition as the piece wise constant function. So a simple measurable function is very useful in measure theory because of the following lemma, because if f is an unsigned measurable function, then there exists a sequence sn of simple measurable functions.

Such that first part is that it is an increasing sequence of non-negative measurable functions and bounded above by f and the second part is that the limit of these measurable functions is precisely the function f. So sn converges to f point wise on x. So this kind of result says that any unsigned measurable functions can be approximated by simple measurable functions.

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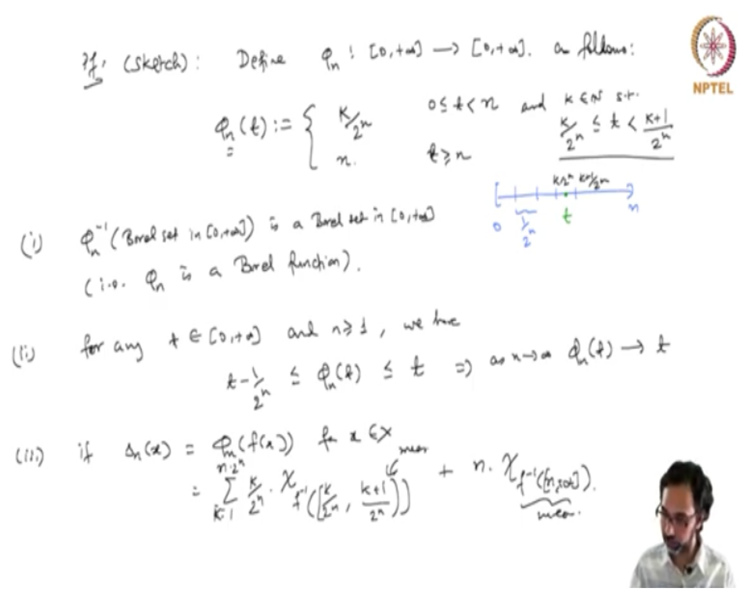

So let us see the proof rather the sketch of the proof. So first of all define functions phi n from 0 to plus infinity as follows. So we define phi n of an extended real number t, if 0 is less than or equal to t is less than or equal to this n, where n is this index here, then we can find a natural number k, such that k over 2 to the power n is less than or equal to t is less than or equal to k + 1 by 2 to the power n and in this case, we write this as k.

So for each t in this interval 0 to n, less than n we find an interval, so we subdivide this interval 0 to n into equal intervals of length 1 over 2 to the power n and our t will lie in one of these intervals and we find this index k such that k by 2 to the power n is less than or equal to t is less than or equal to k + 1 by 2 to the power n. So we are taking this value k here, where t lies between k by 2 to the power n and k + 1 by 2 to the power n.

And then, if t is greater than or equal to n, we just set it as n. So first note that phi n inverse of a Borel set 0 to plus infinity is a Borel set in 0 plus infinity. In other words, phi n is a Borel's function and secondly that for any t and so t in 0 plus infinity and n greater than or equal to 1, we have the inequality t – 1 by 2 to the power n is less than or equal to phi and t is less than or equal to t, which means that as n goes to infinity phi nt converges to the identity function t.

And thirdly that if we set sn of x is equal to phi n of fx for x in x, then this is the sequence of functions that we want. This is a simple function, because it can be written as a sum k = 1 to n times 2 to the power n. Here n times 2 to the power n is the number of subintervals of length 1 by 2 to the power n that we are dividing into for the set 0 to n. So we have k = 1 to n times 2 to the power n of this value k over 2 to the power n, sorry, this should be k over 2 to the power n.

And then you are multiplying by the indicative function of the inverse image of precisely these intervals k by 2 to the power n, k + 1 by 2 to the power n. So here again this should be a strictly less than sign. So these are all Borel sets. So these are all measurable functions. The indicative functions are all measurable functions and plus you have n times chi of the set n + infinity. So this is a finite sum of indicative functions of measurable subsets of x. So this again should be f inverse of n + infinity. So these are all measurable and so this snx is a simple function.

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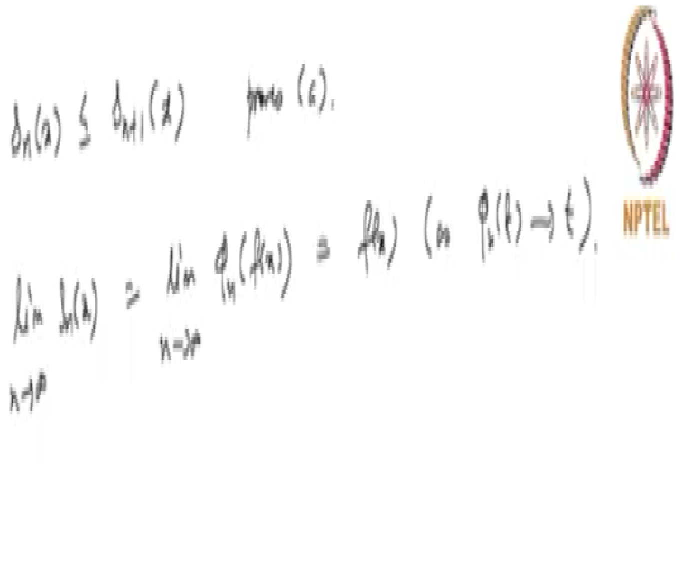

And it is easy to show that snx is less than or equal to sn + 1x. So this proves part a and the limit of snx as n goes to infinity is equal to the limit of n goes to infinity phi n of fx and this is precisely equal to fx as phi n, t converges to the identity function t.