MEASURE THEORY

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Lecture 4
Elementary Sets and Elementary Measure: Part 1

This is just to separate the subsets of $\mathbb{R}^n$ into two groups: one will be the measurable subsets of $\mathbb{R}^n$ and the other is the non measurable subsets. The measurable subsets will follow our geometric intuition about length, area, volume etc. They will also satisfy the finite additivity property. But this property will fail in the non measurable subsets.

Outline of the lecture:

- ➢ Intervals, Boxes and Elementary Sets
- ➢ Elementary measure

Today we will see the notion of elementary sets and elementary measure of those sets. So we will define what are intervals and boxes in higher dimensions, say, in n-dimensional Euclidean space as well as what is an elementary set. Then we will define the elementary measure of those sets and prove that it does not depend on the choice of the description of your elementary set in terms of finite unions of disjoint boxes.

So we can begin our study of measuring the subsets of $\mathbb{R}^n$ from sets which have very simple structure. These are called elementary sets. We will see that we can produce a method to give a numerical value to each such elementary subset of $\mathbb{R}^n$ which not only has the finite additivity property but also it will conform to all geometric intuition via the notion of length, area and volume. So these are called elementary sets and the numerical value that will be assigned to elementary sets will be called elementary measure. So let us begin our first case for the real line $\mathbb{R}$. Let us consider those subsets of $\mathbb{R}$ from which we

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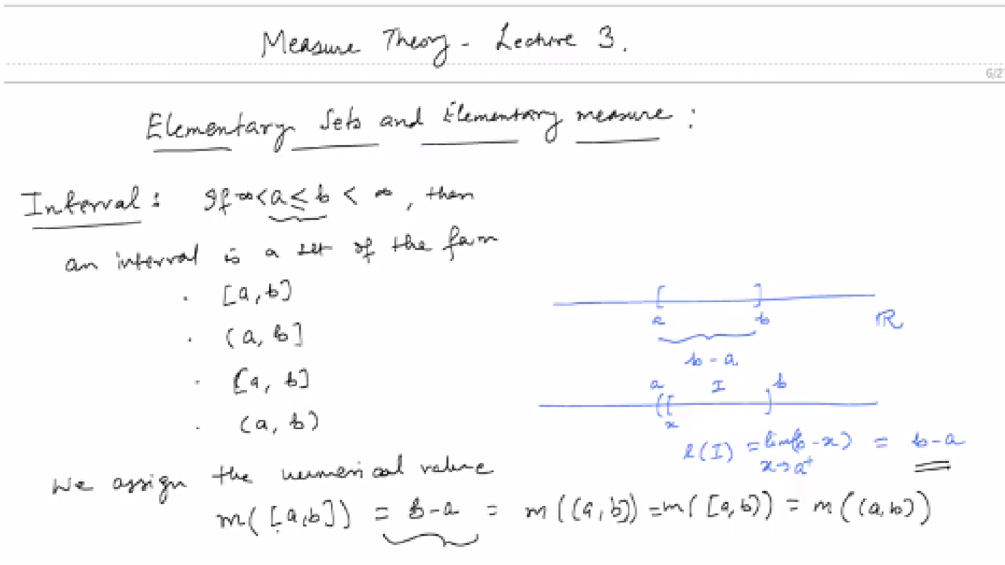
can easily determine what should the length of such a subset. The easiest subset that one can think of is in fact the interval (a, b), say.



So of course we know that the length of the interval is $(b - a)$. Now we can try to see if we have rather an open end at a and the end at b is closed, that is, consider the interval $I = (a, b]$, then the length is

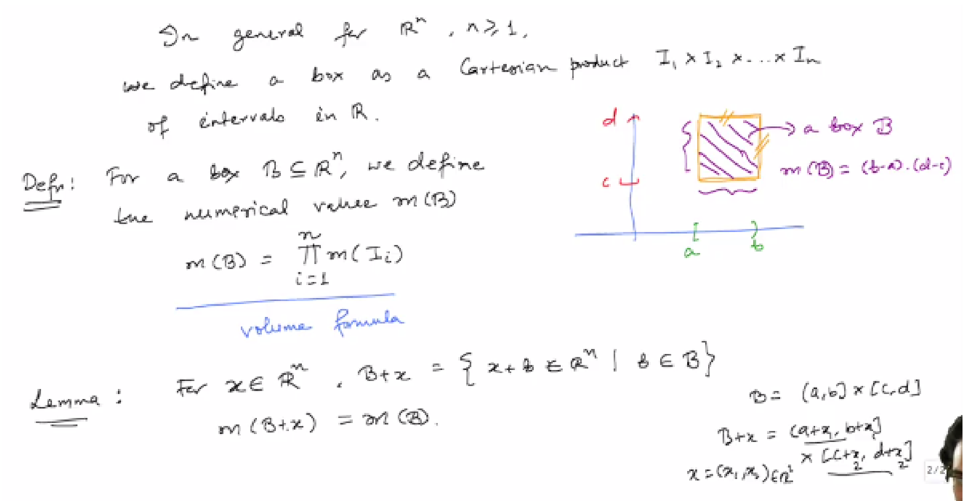
$$\ell(I) = \lim_{x \to a^+} (b - x) = b - a.$$

Therefore we can consider all kinds of intervals which are either both endpoints closed, or one of them can be open, or even both of them can be open, and the lengths will still be the difference between the higher endpoint and the lower endpoint. So we make this definition for an interval.

Interval: If $-\infty < a \leq b < \infty$ (having considered also the degenerate case when it is just one point and in this case our length will just be zero), then an interval is a set of the form: $[a, b]$,$(a, b]$,$[a, b)$ and (a, b). So any such subsets of $\mathbb{R}$ will be called an interval in our terminology. We assign the numerical value

$$m([a, b]) = b - a = m((a, b]) = m([a, b)) = m((a, b)).$$

So in all such cases we will assign the value $(b - a)$ to such an interval. So for all these cases, the numerical value is simply $(b - a)$.



In higher dimensions, in general, for $\mathbb{R}^n$, $n \geq 1$, we define a box as a Cartesian product $I_1 \times I_2 \times \cdots \times I_n$ of intervals in $\mathbb{R}$. For example, in dimension two, if our one of our intervals is (a, b) and the other interval is (c, d), then our box will be simply the square with appropriate sides removed whenever we encounter an open endpoint.

So in this case our b endpoint is open, so this side will be open and our d endpoint is open, so this side will also be open. So any such region enclosed by such lines will be called a box. So just from the general meaning of the term we call it a box.

Definition: For a box $B \subseteq \mathbb{R}^n$, we define the numerical value

$$m(B) = \prod_{i=1}^{n} m(I_i) \text{ (Volume formula).} \tag{1}$$

So, here for this box B in the picture, we will have the measure of B to be simply $(b - a)(d - c)$. So it is just the product formula for the area in this case. Similarly in higher dimensions, (1) is the volume formula for a box.

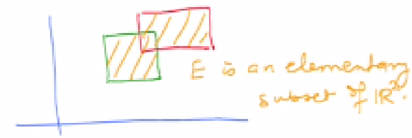
So, now we have started with the most elementary notion of a subset which has an obvious size which is given by the length or the area or the volume.

Lemma 0.1. *For $x \in \mathbb{R}^n$, $B + x = \{x + b \in \mathbb{R}^n : b \in B\}$*

If $B = (a, b] \times [c, d]$, then $B + x = (a + x_1, b + x_1] \times [c + x_1, d + x_2]$ for $x = (x_1, x_2) \in \mathbb{R}^2$. It is clear that the length of each of the later intervals remains invariant because we are translating it by an amount $x = (x_1, x_2)$. So the first interval is translated by an amount of x_1, and the second interval is translated by an amount x_2. But the resulting length remains the same. Now we can define what is an elementary set in $\mathbb{R}^n$.

Defn: (Elementary Set in $\mathbb{R}^n$): A subset $E \subseteq \mathbb{R}^n$ is called elementary if it is a finite union of boxes.

Thm: Let E be an elementary subset of $\mathbb{R}^n$. Then



(i) E can be expressed as a finite union of disjoint boxes.

(ii) If E is a union of disjoint boxes $\{B_i\}_{i=1}^{n}$ and E is also a union of disjoint boxes $\{B_j'\}_{j=1}^{n'}$, then

$$m(E) := \sum_{i=1}^{n} m(B_i) = \sum_{j=1}^{n'} m(B_j')$$

$m(E)$ is called the elementary measure of E.

Definition 0.2. *(Elementary set in $\mathbb{R}^n$): A subset $E \subseteq \mathbb{R}^n$ is called elementary elemetary if it is a finite union finite union of boxes.*

Now we have the following theorem:

Theorem 0.3. *Let E be an elementary subset of $\mathbb{R}^n$. Then*

(1) *E can be expressed as a finite union of disjoint boxes.*
(2) *If E is a union of disjoint boxes $\{B_i\}_{i=1}^{n}$ and E is also a union of disjoint boxes $\{B_j'\}_{j=1}^{n'}$, then*

$$m(E) := \sum_{i=1}^{n} m(B_i) = \sum_{j=1}^{n'} m(B_j').$$

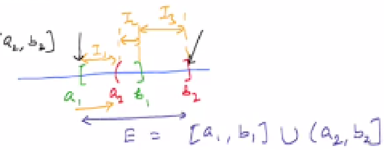
Here $m(E)$ is called the elementary measure of E.

The first one says that E can be expressed as a finite union of disjoint boxes. So our elementary subsets may have boxes which are overlapping. But the first part says that it can always be expressed as a finite union of disjoint boxes.

Secondly, it says that it does not matter what kind of partitioning of E you consider in terms of finite union of disjoint boxes. So it is important to have disjoint boxes and then regardless of the partitioning, this sum of the measures of the each box is will be same. This common value will be denoted as $m(E)$.

Pf: (i) To show: An elementary set E is a finite union of disjoint boxes.

$n = 1$: let $E = [a_1, b_1] \cup (a_2, b_2]$



then $E = \underbrace{I_1 \cup I_2 \cup I_3}$

$a_1 \leq a_2 \leq b_1 \leq b_2$

$\longrightarrow$ increasing order

$$\begin{cases} I_1 = [a_1, a_2) \\ I_2 = [a_2, b_1) \\ I_3 = [b_1, b_2] \end{cases}$$

Generalize to k-intervals:

$E = \bigcup_{k=1}^{n} I_k$ (may not be disjoint)

we can arrange the end-points of the intervals I_k's ($2k$-endpoints) in increasing order:

$a_1 \leq a_2 \leq a_3 \leq \dots \leq a_{2k}$. (increasing order)

So let us prove the first part of the theorem. We have to show that an elementary set E is a finite union of disjoint boxes. Let us begin with the simplest case, that is, for $n = 1$. So the dimension of $\mathbb{R}^n$, where n, the dimension of Euclidian space is one. So we have the real line $\mathbb{R}$ and E is an elementary subset of $\mathbb{R}$. Suppose that E can be written as a union of two boxes. So the first one, say, a closed interval $[a_1, b_1]$ and the second one is $(a_2, b_2]$. So there is some overlap. Now we would like to partition this. So this is the union, the whole region will be the our set E. E is the union of the interval $[a_1, b_1]$ and $(a_2, b_2]$. So the obvious thing to do is to have the first interval. The first interval should be this one I_1. The second interval should be this one. This is I_2 and the third interval should be the rest I_3. So what we have done here is that we have taken the endpoints of the these intervals and we have arranged them in an increasing order from left to right. So $a_1 \leq a_2 \leq b_1 \leq b_2$. So you can even have the endpoints to coinside. So that is why I have put an less than or equal to sign. Of course, here, there in our example, there is a strict inequality. But, in general, the endpoints can be the same. So we have this. We have arranged the endpoints of the intervals in increasing order and then we can define our intervals $I_1 = [a_1, a_2)$, $I_2 = [a_2, b_1)$ and $I_3 = [b_1, b_2]$.

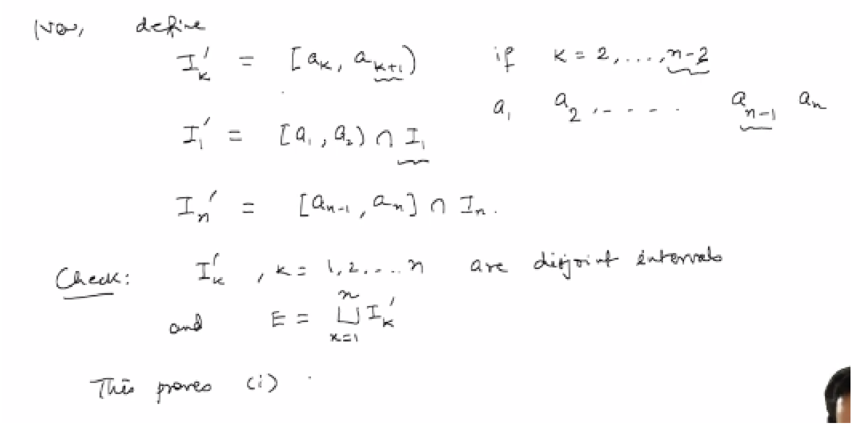
We have whatever whether it is closed or open, we put that condition for the first one and for the last one. Whether it is closed or open, we also put the same. So now we can write our E as the disjoint union of I_1, I_2 and I_3. So this simple geometrically obvious method works for more than one intervals and also in higher dimensions.

So this was for our baby example $[a_1, b_1] \cup (a_2, b_2]$. Then this is our decomposition, where I_1, I_2, and I_3 are given by this this expression. So now we can easily generalize to k intervals in the dimension one. So

$E = \cup_{k=1}^{n} I_k$. But now, our I_ks may not be disjoint. So this union may not be a disjoint union. But, again, we can arrange the endpoints of the intervals I_k's. So we will have $2k$ endpoints. We have k intervals and each has two endpoints. So we have, in total, $2k$ endpoints in increasing order:

$$a_1 \leq a_2 \leq \cdots \leq a_{2k}.$$

So we have $2k$ points which are now arranged in increasing order. Now we are going to do the same thing that we did in this example.



Now, define

$I_k' = [a_k, a_{k+1})$ if $k = 2, \dots, n-2$

$a_1 \quad a_2, \dots \quad a_{n-1} \quad a_n$

$I_1' = [a_1, a_2) \cap I_1$

$I_n' = [a_{n-1}, a_n] \cap I_n$.

Check: I_k', $k = 1, 2, \dots n$ are disjoint intervals

and $E = \bigsqcup_{k=1}^{n} I_k'$

This proves (i)

So now define $I_k' = [a_k, a_{k+1})$ if $k = 2, \dots, n-2$. Here we are left with the least endpoint and the highest value endpoint. When you have $n-2$ and you do a_{k+1}, then you end up with a_{n-1}. So the intervals with these endpoints are covered. What is left is a_1 and a_n. So I define

$$I_1' = [a_1, a_2) \cap I_1$$

Here I am taking the intersection because I_1 may be open at the point a_1. So then you will end up with the open interval. Similarly, we define

$$I_n' = [a_{n-1}, a_n) \cap I_n.$$

So I leave it as an exercise that $I_k', k = 1, \dots, n$ are disjoint intervals and E can be written as the union of these disjoint intervals I_k', $k = 1, \cdots, n$. So this proves our assertion in the case $n = 1$, that is, this proves the first part of the theorem for $n = 1$.