Measure Theory
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Lecture – 39
Measurable Functions: Definition and Basic Properties – Part 1

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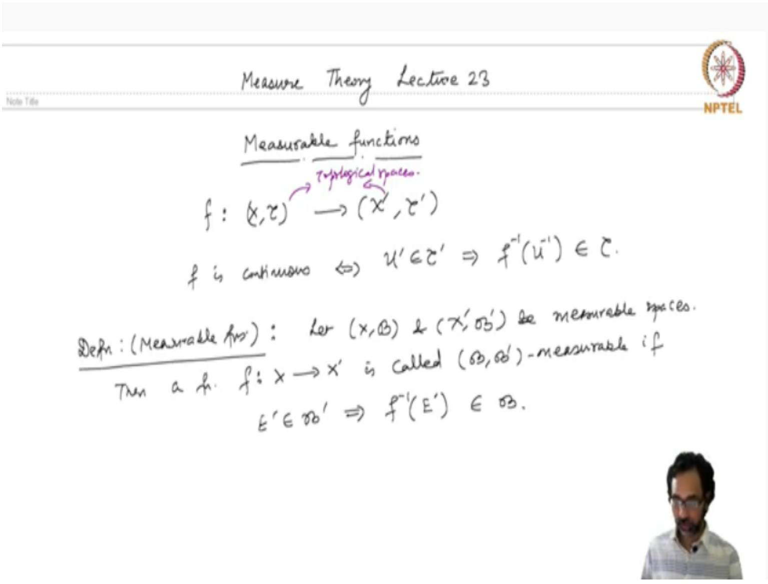

We now come to a new topic which is that of measurable functions. So, we know that if f is a function between two topological spaces x tau and x prime tau prime. So, these two are topological spaces then f is continuous f is said to be continuous if and only if any open set in x prime u prime if you have any open set u prime then f inverse of u prime is open in x. So, f is continuous if and only if pre images of open sets are open so in the same way we make the following definition.

So this is about measurable functions between two measurable spaces so let x, B and x prime B prime be measurable spaces meaning that B and B prime are sigma algebras on x and x prime then a function f from x to x prime is called B, B prime measurable if you take any set E prime in B prime then this implies that the pre image of E prime via f belongs to the sigma algebra B. So, this is very much like the definition of a continuous function except that instead of typologies we have sigma algebras.

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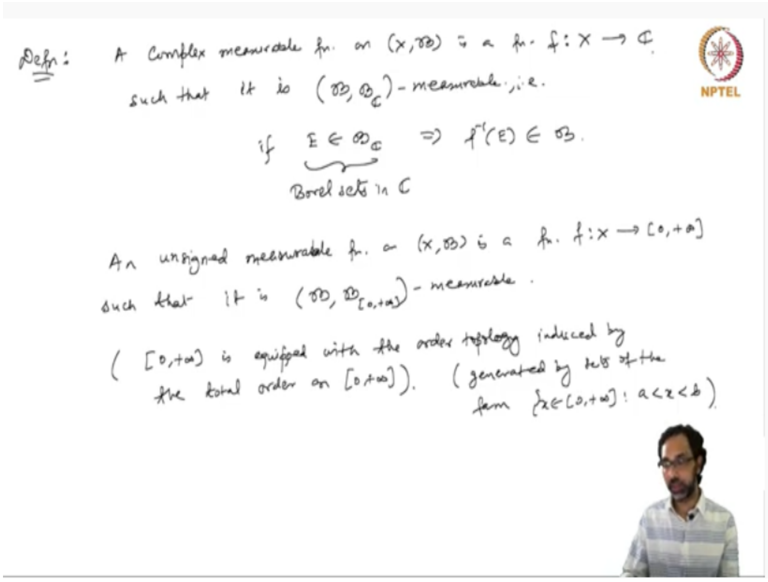

Now so this is again a definition a complex measurable function on a measurable space x, B these a function f which takes inputs in x and gives you complex numbers such that it is B and you take the Borel sigma algebra on c measurable. So, we are equipping c with the Borel sigma algebra which makes it a measurable space and so f is called a complex measurable function if it is B, B c measurable meaning that if E belongs to B it is a Borel subset of c.

So, elements of the Borel sigma algebra are called Borel subsets these are called Borel subsets or simply boral sets in c. So, if you take a Borel set in c this implies that f inverse of E belongs to this sigma algebra B on your measurable space x. Similarly, an unsigned measurable function on x, B is a function again from x but this time it takes non-negative extended real numbers such that it is again you take B on the for x and again you take the Borel sigma algebra for subsets Borel subsets of the non-negative extended real numbers.

Since we are talking about Borel subsets of the extended real numbers we have to specify what kind of topology we are putting on the extended non-negative real numbers. So, this is a 0 plus infinity so is equipped with the order topology on 0 infinity so well-ordered topology induced by the total order on 0 infinity. So, the order topology is defined for any totally ordered set and it contains open intervals of the form so its generated by this is generated by sets of the form a, x belong to plus infinity says that a < x < b. so, here a and b can be any real number as well as plus infinity.

So, this gives you a sub base of the typology and then you can generate a topology on the non-negative real lines. So, in this way we have defined what is a complex measurable functions and then unsigned measurable function. So, we will be interested in both these kinds of functions and note here that we have not considered Lebesgue measurable sets for the or we are not considered Lebesgue sigma algebra for the space c the complex numbers or even 0 plus infinity this is because.

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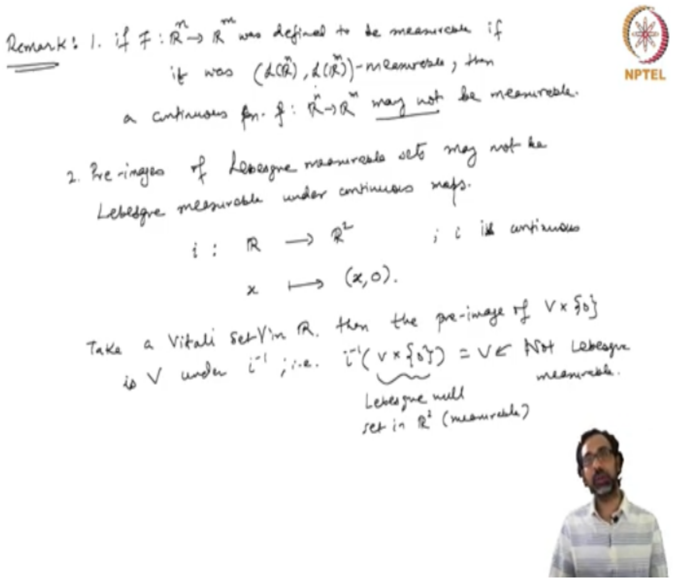

So I will put this as an remark and it is important to highlight this fact that if you wanted to simply mimic the definition of a continuous function and if you had if F from R to R was defined to be measurable to be measurable if it was L R, L R measurable so we would take not the Borel but the Lebesgue sigma algebra on both sides then a continuous function f may not be measurable.

So, our definition is a bit asymmetric in the sense that we only considered the Borel sigma algebra on the target space and not the Lebesgue sigma algebra precisely because we want continuous functions to be measurable and very easy examples show that the pre images of Lebesgue measurable sets under continuous maps may not be Lebesgue measurable. So pre images so this is the first remark secondary mark is that pre images of continuous of Lebesgue measurable sets may not be Lebesgue measurable under continuous maps.

So, it is not difficult to give an example for this so let me write rather we can consider R rather than considering functions from R to R you can consider functions from Rn to Rm and we can put the Lebesgue sigma algebras on Rn and Rm respectively here again Rm to Rn. So, in this case also pre images of Lebesgue measurable sets under continuous maps may not be the Lebesgue measurable.

So, this also works for r to r but let me write it for Rn to Rm because this example is much simpler. So, suppose that you have a map from R to R2 which simply takes x to the point x, 0. So it is an inclusion of R in R2. So, this inclusion map is of course continuous is continuous and if you would now take non-Lebesgue measurable set let me take the vitali set in R then the pre image of so let me call this vitali set V then the pre image of V cross 0 is V under I inverse meaning that i inverse of V cross 0 is simply V because it is an inclusion map.

On the other hand, V cross 0 this is a Lebesgue measurable in fact a Lebesgue null set in R2 with respect to the Lebesgue measure of R2 and so measurable Lebesgue measurable but V is not Lebesgue measurable. So, we see that even very simple continuous functions can give you the pre image of Lebesgue measurable sets may not be Lebesgue measurable under such continuous maps.

So, because we want our continuous functions to be measurable, we have this condition that rather than taking the Lebesgue sigma algebra on the target space we only consider the Borel sigma algebra and then the premier of a Borel sigma measurable set is always Lebesgue measurable. So then there is no problem and we have that continuous functions are in fact measurable.

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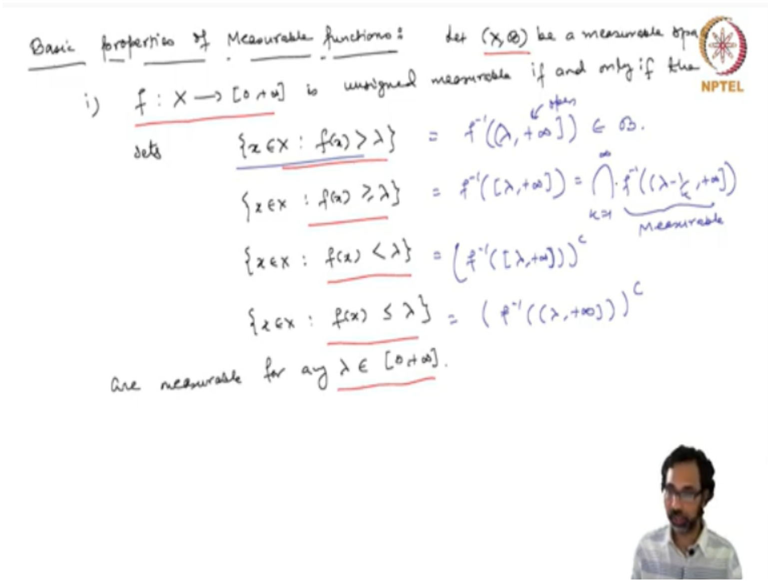

So now let us come to some basic properties of a measurable functions so we start with the measurable space x, B and we take an unsigned measurable function f an unsigned function meaning that the range is the non-negative extended reals. So, this first property says that it is measurable if and only if these four kinds of sets x in x such that greater than lambda greater than equal to lambda less than lambda and less than equal to lambda where lambda is any non-negative real number.

These are all measurable so rather than giving a proof I will just a mention here in one line what are the reasons? So, this is equal to f inverse of the set the open interval lambda plus infinity and so this is open and so this is measurable. Similarly, we can show that this is precisely the set lambda plus infinity and this can be written as this is written as the intersection of the sets f inverse of open intervals lambda minus 1/K to plus infinity included where K is from 1 to infinity.

So, this is these are all open sets and so these are measurable and so these are this is a Borel subset sorry these are all measurable sets so in the intersection is also measurable because B is a sigma algebra. So similarly, one can show this for fx less than lambda this is f inverse of lambda plus infinity compliment and this one is f inverse open lambda plus infinity compliment.

So, because these closed under compliments we have that these are all measurable. So, the other way on the other side we have to show that if these one of these is measurable then let us say the first one for the first one if this is measurable then f inverse of any Borel set is an element of the sigma algebra. So, this shows what we have done here is just shows that if f is unsigned measurable then these are measurable sets. On the other end, we have to show that if these are measurable then it is unsigned measurable.

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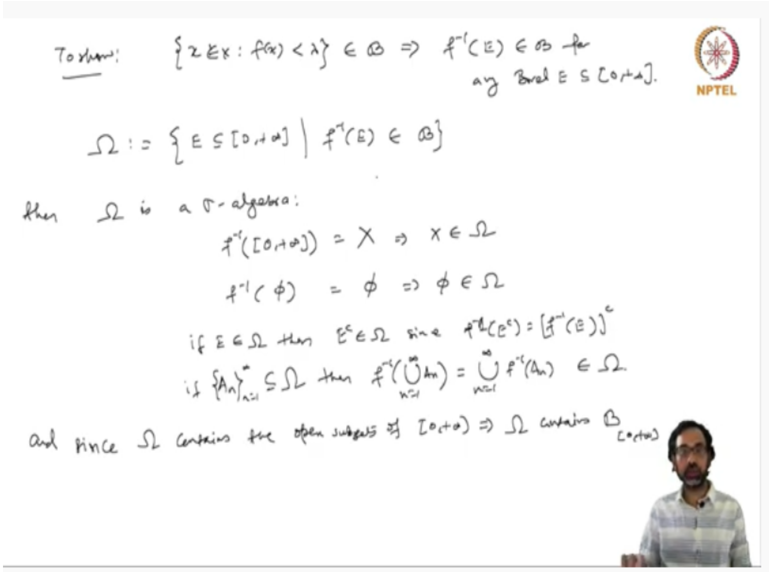

So, to show that if x in x such that fx let us say I will take less than lambda this one is measurable this implies that f inverse of E belongs to B for any Borel set E in 0 plus infinity. So, this can be shown using the fact that if you let omega to be the collection of subsets of 0 plus infinity such that f inverse e belongs to B then I claim that omega is a sigma algebra because f inverse of the whole interval plus 0 to plus infinity is the whole space x.

So, x belongs to omega f inverse of the empty set is empty. So, this implies that the empty set belongs to omega if E belongs to omega then E compliment also belongs to omega since f inverse of E compliment = f inverse of E compliment. So, it is closed under compliments and if A ends a collection in omega then f inverse of the union n = 1 to infinity n this is equal to the union of the individual pre images n.

So, this belongs to omega so this is this shows that omega is a sigma algebra and since now omega contains the open sets of 0 minus infinity this implies that omega contains the Borel subsets Borel sigma algebra as well and so if you have it is enough to have a measurability for open subsets to get measurability for Borel subsets. So, we have shown that this is an if and only if condition.

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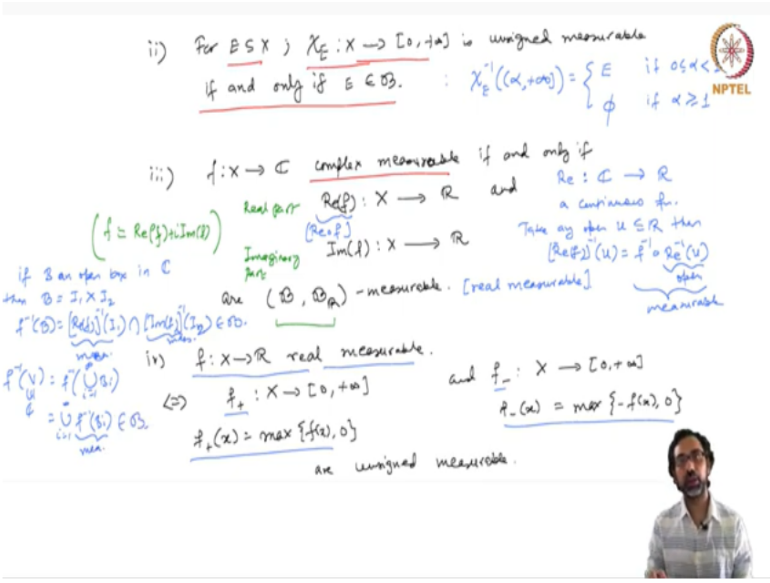

Now we have three more important properties so the first one says that if E is a subset of x and the chi E is the indicator function then it is unsigned measurable if and only if E itself is a measurable subset of x. So, the proof is quite simple and we can simply take the inverse image of an open interval alpha to plus infinity included and note that if 0 less than or equal to alpha strictly less than 1 then this interval contains 1 but does not contain 0.

So, you will have the inverse image should be simply E and if alpha is greater than or equal to 1 then this interval does not contain 1 or 0 and so this is going to be the empty sets. So, in both cases this is measurable and so of course if E is measurable then the inverse images of all these sets are measurable and if these are measurable then E is measurable. So, in both ways we get that this integrative function is unsigned measurable if an only if E is measurable.

On the other hand, if you have a complex measurable function then the real part so this one is the real part of f and this one is the imaginary part of f both are functions from x to now the real numbers and these are measurable with respect to the sigma algebra B and the again the Borel sigma algebra on R. So here f is written as Ref minus Inf times the imaginary complex number I. So, in this case the real part and imaginary part are measurable with respect to the Borel sigma algebra on R and the given sigma algebra B on x.

So, in this case we also call it real measurable function. So to show this it is enough to note the following that the real part is itself a function from the complex numbers to the real numbers and it is a continuous function from c to R and so if you take any open set u in R then the inverse image under the real part of f of u is nothing but so you can view this as a composition so there is real part of f is a composition of two functions the real part and f itself.

So, the inverse is just the composition reversed so f inverse composed with r inverse of u and because the real part is a continuous function this is open and so this is an inverse image of an open set so this is measurable so we get that the real part is measurable and simply the imaginary part is also measurable. On the other hand, if x is now a real measurable function then the positive part f plus which is defined as the maximum of fx and 0 and the negative part they are unsigned measurable and this is again an if and only if condition.

So, we see that we have various equivalent conditions in terms of for f to be complex measurable and for f to be real measurable and for an indicator function of a subset of x to be unsigned measurable. On the other hand, if the real and imaginary parts are real measurable then we can conclude that f is complex measurable because we have if B is an open box in the complex plane open box in c then you can write B as a product of two intervals and then f inverse of B is precisely real part of f inverse of I 1 intersection with the universe of the imaginary part for I 2 and these two are measurable.

So, f inverse B belongs to the sigma algebra B when you have an open box now if you have any open set V open subset of the complex plane then you can write V as the union of open boxes I = 1 to infinity need not even be an open box it just needs to be a box and this is the union of f inverse of the Bi and these are measurable so this belongs to the sigma algebra B.