Measure Theory
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Module No # 08
Lecture No # 38
Examples of Measures defined on Rd via Hahn Kolmogorov extension – Part 2

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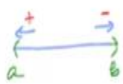
Lebesgue-Stieltjes elementary measure:

For any $a \in \mathbb{R}$, we have

$$\mu_F(\{a\}) := F_+(a) - F_-(a).$$

For $-\infty < a < b < +\infty$; define



$$\mu_F((a,b)) := F_-(b) - F_+(a).$$

$$\begin{aligned}\mu_F([a,b)) &:= \mu_F(\{a\}) + \mu_F((a,b)).\\ &= \cancel{F_+}(a) - F_-(a) + (F_-(b) - \cancel{F_+(a)})\\ &= F_-(b) - F_-(a).\end{aligned}$$

Now let me give another construction of a measure for the Lebesgue Stieltjes measure on the real line which uses the Hohn Kolmogorov extension theorem. So for this we define what is called a monotonically non-decreasing function. So a function f from R to the positive real rather extended positive real is called monotonically non-decreasing if Fx is less than equal to Fy whenever x is less than or equal to y.

So it is a monotonically non-decreasing function so define for a monotonically non decreasing function for F monotonically non-decreasing. The following 2 quantities which is for x in R we define F –x to be the supremum of values Fy where y is less than x strictly less than x and F + x is the infimum for values greater than xy greater than x for the values Fy. And we have F – x is less than or equal to Fx is less than or equal to F + x because F is monotonically non-decreasing.

So using this function F we will define a premeasure by giving a formula for the premeasure on bounded intervals.

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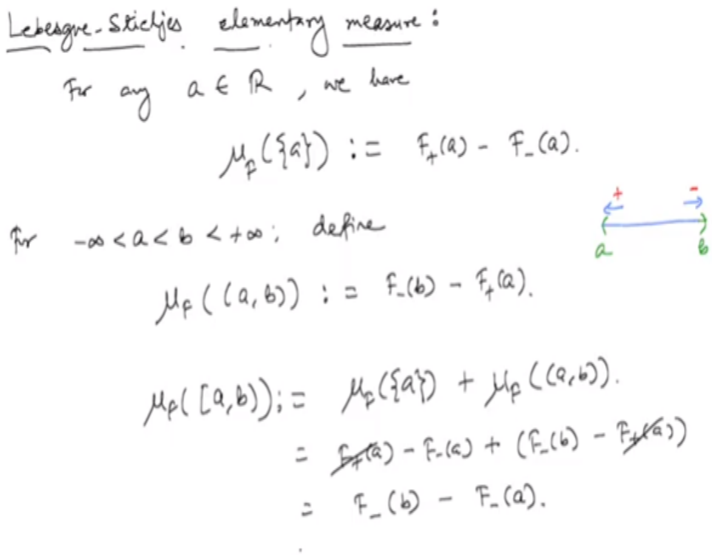

Now we can define what is called the Lebesgue Stieltjes elementary measure. So this is defined as follows for any number real number a, we have. So I will denote this as Mu f of the singleton set a this is by definition F + a – F– a. So, in contrast with the elementary measure where we had the elementary of the singleton sets where 0 but here they may not be 0.

So the Lebesgue Stieltjes elementary measure may give a non-zero finite weight to a singleton set. Now using this we can define for a finite numbers a and b define for the open interval a, b this is by definition because this is the open interval a, b. So we would like to approach a from the right and approach b from the left so then this Lebesgue Stieltjes elementary measure is given by F – b because we are approaching it from the negative lower values of b and we are approaching it from the higher values of a so there should be a + sign and so –F + a.

So this is the formula for the Lebesgue Stieltjes elementary measure for an open interval and now we can reduce for any other interval which contains the end points which is simply by adding the value of the singleton set. So Mu F of a, b this is equal to Mu f of a+ again by definition Mu F of a, b and this is F + a – F – a + F – b – F+ a. So this gives us the value F – b so F + a, and –F + a cancel out so F – b – F – a.

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$$\mu_F((a,b]) := \mu_F((a,b)) + \mu_F(\{b\})$$
$$= (\cancel{F_-(b)} - F_+(a)) + (F_+(b) - \cancel{F_-(b)})$$
$$= F_+(b) - F_+(a).$$

$$\mu_F([a,b]) := \mu_F\{a\} + \mu_F((a,b)) + \mu_F(\{b\})$$
$$= [\cancel{F_+(a)} - F_-(a)] + (\cancel{F_-(b)} - \cancel{F_+(a)}) + (F_+(b) - \cancel{F_-(b)}).$$
$$= F_+(b) - F_-(a)$$

Also, $F_+(+\infty) = +\infty$; $F_-(-\infty) = -\infty$.

$F_-(+\infty) = \sup_{y \in R} F(y)$; $F_+(-\infty) = \inf_{y \in R} F(y)$.

Similarly we can do it for other cases Mu F of a, b with end point b included this is going to be Mu F a, b + Mu F of the singleton set b and this is nothing but again F – b – F + a + F + b – F – b and then F – b cancel out. So you get F + b – F + a so one can simply try to memorize this formulas but it is better to just memorize 2 cases. First is that one for the singleton set this formula this is for the singleton set and for the open set. And we have the idea that if you have open set the approach from each end points from within the interval.

So the higher end point you approach from the left and the lower end point you have to approach from the right. So conceptually if you understand these 2 cases then you can derive all the rest of the formulas using some kind of finite additivity law imposed on these elementary measures. So then I will just write down one more case which is when both end points are closed. So this is again Mu F a + Mu F open interval a, b + Mu f of the singleton set b.

So this is F + a – F – a + F – b – F + a + F + b – F – a so many things will cancel out for example F + a, and F + a cancel and then. So this should be F – b here and then here F – b and F – b cancel out then you are left with F + b – F – a. So therefore we can derive all this formulas for the elementary Lebasgue Stieltjes elementary measure. Now we can extend the formula for the Lebasgue Stieltjes elementary measure to half infinite and in infinite interval inverse of the form.

So first also we have F + of + infinity is equal to +infinity F- of minus infinity equals minus infinity and F – of +infinity. So there is no set here it is just + infinity and –infinity and here F-of plus infinity is the supremum because we are approaching from the infinity from the left. So it is the supremum of all y of values Fy and when you approach minus infinity from the right then you should get the infimum of all values Fy.

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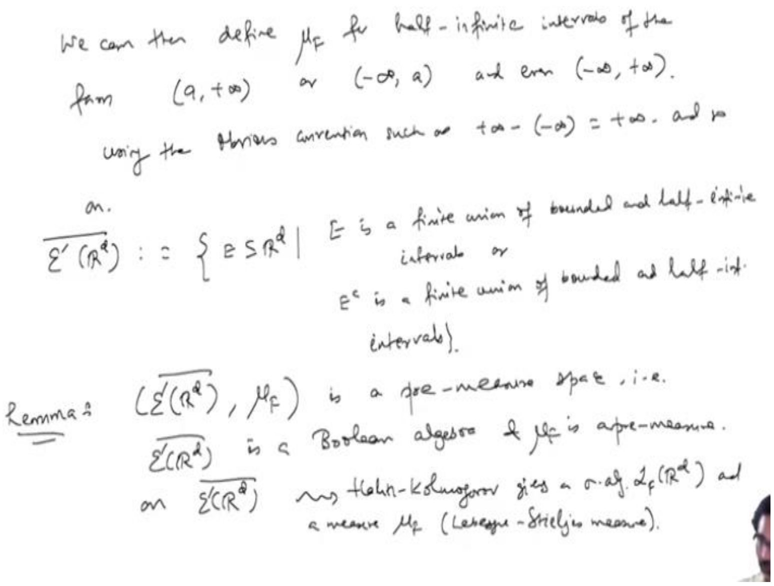

So using this we can then define Mu F for half open rather half infinite we can then define Mu F for half open rather half infinite intervals of the form a + infinity or minus infinity a and so on. And even minus infinity plus infinity so we can use exact same formulas we have used before just by taking this values for F – F + on plus and minus infinities and then using the obvious convention such as if you take such as plus infinity minus infinity is going to be plus infinity and so on.

So we have defined it for bounded intervals as well as unbounded intervals now we can define the elementary subsets of Rd these are finite unions. So E subset of Rd such that E is a finite union of bounded and half infinite intervals or the complement is the finite union of bounded and half infinite intervals. So this gives us the new Boolean algebra so one can check that this is a Boolean algebra so I will write Lemma and leave this as an exercise for you to show Lemma that Ef Rd well this Boolean algebra does not really depend on F.

So E bar Rd is with the measure Mu F is the premeasure space meaning that this is the Boolean algebra and Mu F is a premeasure on this Boolean algebra E Rd. So the difference between this one I should change the notation because this is not quite the elementary algebra as we have defined it. So let me write E prime throughout so but still one can try to compare E prime with the elementary algebra E.

Nevertheless you have a premeasure Mu F on this new Boolean algebra and so the Hahn Kolmogorov extension theorem gives a measure a sigma algebra, which let me call it Lf Rd and measure which are continue to call Mu F which is called the Lebesgue Stieltjes measure on Rd,

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[Ex]: if F = identity map then.

$\mu_F = m$ (Lebesgue measure).

So in particular if particular example of a Lebesgue Stieltjes measure is if F is equal the identity map. Then this Mu F is nothing but the Lebesgue measure so all this are the exercises for you to check that first this the Lemme says that this is a Boolean algebra and a premeasure and then the Hahn Kolmogorov thing gives you a Lebesgue Stieltjes measure and for the particular case of the identity map this gives you the Lebesgue measure.

So this one subtlety here is that I defined it for a monotonically non-decreasing function which takes positive extended positive real values but I should actually try to replace this by r. So, if you have a monotonically real value non-decreasing function then you can define the whole thing and the identity function is then admissible here as a monotonically non-decreasing function. So then you will still get a Mu F here is taking values in the positive extended real numbers because F is monotonically non-decreasing.

So F Mu f on intervals or half infinity so finite intervals or half infinite intervals give you a value between 0 and plus infinity. So it is a first one is to check that it is a finitely additive measure then one has to check that it is a premeasure and then for the particular case of the identity one should check that you get nothing else but the Lebesgue measure.