Measure Theory
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Module No # 08
Lecture No # 37
Examples of Measures defined on Rd via Hahn Kolmogorov extension – Part 1

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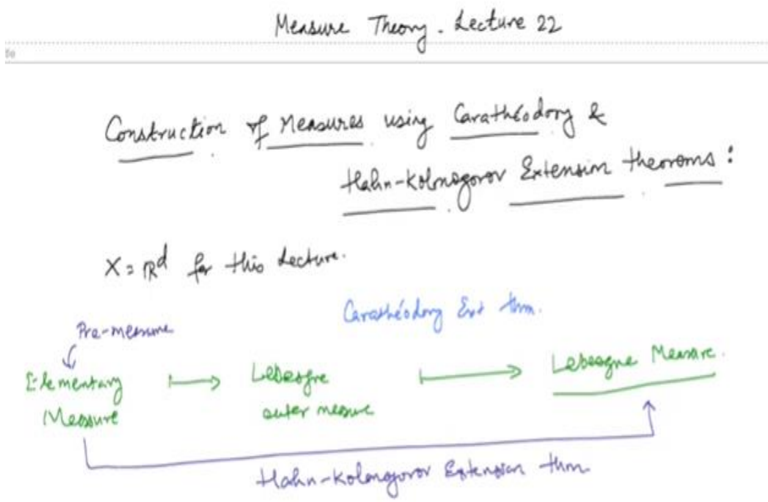

We have seen how the Caratheodory extension theorem Hahn Kolmogorov extension theorems can be used to upgrade an existing outer measure or an existing pre measure to a full measure and using this prescription. We can actually give many examples of new measures on the Euclidean space Rd and also other spaces. But for this lecture we will restrict ourselves to Rd so x = Rd for this lecture.

And we will see how the Hahn Kolmogorov extension theorem actually gives you the Lebesgue measure we have seen from the Caratheodory extension theorem. We get from Lebesgue outer measure we got the Lebesgue measure we proved in the last class that the Caratheodory measureable subsets are exactly the Lebesgue measureable subsets. So the restriction of the Lebesgue outer measure to both this sigma algebras is the same and this is the Lebesgue measure.

But in fact remember that we had a definition of Lebesgue outer measure using the elementary measure and it claimed in my last lecture that this passage from the elementary measure to the Lebesgue measure is given by the Hahn Kolmogorov extension theorem. So in this claim it is implicit that the elementary measure should be a premeasure this should be a premeasure on a Boolean algebra.

And the Boolean algebra we will take is the elementary algebra. So if we prove that the elementary measure is the pre measure on the elementary algebra then using the Hahn Kolmogorov extension theorem it is not difficult to show that one gets exactly the Lebesgue measure when you apply the Hahn Kolmogorov extension theorem.

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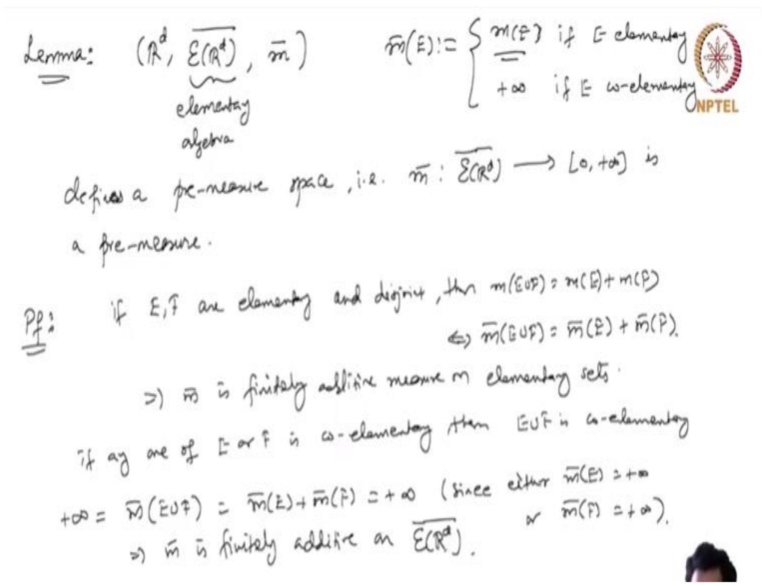

So let us see first that this Lemma that Rd equipped with the Boolean algebra ERd bar so this was the elementary algebra of elementary or co-elementary sets. And equipped with this M bar which was the extended elementary measure given by mE if E is elementary and plus infinity if E is co-elementary. So this gives us is defines a pre-measure space meaning that m bar from this elementary algebra to this non-negative extended real’s is a pre measure.

So let us prove this so we already know that the elementary measure on elementary subsets is finitely additive. So if E and F are elementary and disjoint then m of E union F equals mE + mF and this is the same as saying that m bar of E union F = m bar of E + m bar of F. So this implies that m bar is finitely additive measure on elementary sets. Similarly if any one of E or F is co-elementary then E union F is co-elementary and so we get plus infinity which is the measure of E union F and this is not m bar E + m bar F this is also plus infinity.

Because either m bar is plus infinity or m bar F is plus infinity so this means that m bar is finitely additive on E bar of Rd. So you have a finitely additive measure on the Boolean algebra of elementary and co-elementary sets.

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Claim: $\bar{m} : \overline{\mathcal{E}(\mathbb{R}^d)} \longrightarrow [0, +\infty]$ is a pre-measure.

($\Leftrightarrow$) if $\{E_n\}_{n=1}^{\infty} \subseteq \overline{\mathcal{E}(\mathbb{R}^d)}$ s.t. $\bigcup_{n\geq 1}^{\infty} E_n \in \overline{\mathcal{E}(\mathbb{R}^d)}$ then

(disjoint)

$\Rightarrow \bar{m}\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \bar{m}(E_n)$

(a) if all E_n's are elementary then $\bigcup_{n=1}^{\infty} E_n$ can be either elementary or co-elementary.

Eg. if $E_n = [\frac{1}{n}, 1-\frac{1}{n}]$ $\Rightarrow \bigcup_{n=1}^{\infty} E_n = (0, 1)$. [check].

$\Rightarrow$ countable union of elem. sets can be elementary.

if $E_n = [-n-1, -n) \cup [n, n+1]$, $n \geq 1$ $\Rightarrow \bigcup_{n\geq 1}^{\infty} E_n = (-\infty, -1] \cup [1, \infty)$

$\Rightarrow$ countable union of elem. sets can be co-elem. $\Rightarrow \left(\bigcup_{n=1}^{\infty} E_n\right)^c = (-1, 1)$.

So now let us prove that m bar is a premeasure which means that if you have a countable infinite collection of the elementary algebra such that the union itself is either elementary or co-elementary then you should have if in addition they are disjoint. So one should also assume the disjointness then this implies that the measure of the union is the sum of the individual terms. So this was our definition of a premeasure.

So now let us analyze what kind of unions are possible so let us divide them in cases if all En's are elementary then union of En n =1 to infinity can be either elementary or co-elementary. So this is an observation while the elementary or co-elementary. So how can we show this so we are just going to give example so example if En is given by let us say 1 / n and 1 – 1 / n. So this implies that the union of En's n = 1 to infinity is nothing but the open intervals 0, 1 so one should check this that you get the union of the open interval 0, 1 so it is elementary.

So in this case you have countable union of elementary sets can be elementary so countable union of elementary sets can be elementary. And of course if you have a countable union you can make it a countable disjoint union and these sets will still be a elementary and so even a countable disjoint union of elementary sets can be elementary. Now if you take En to be –n-1, -n union n, n+1 then this is an elementary set but the union of En’s n = 1 to infinity.

So here n is greater than or equal to 1 so the union of En’s is nothing but minus infinity to -1 union 1 to infinity. And so this is a co-elementary set because this complement is nothing but the set -1, 1 which is elementary. So this means that countable union of elementary sets can also be co-elementary. So, countable union of elementary sets can be co-elementary. So, in both cases we have to show that the when you have disjoint union of elementary sets which are either elementary or co-elementary. Then you have this relation which is the countable additivity relation.

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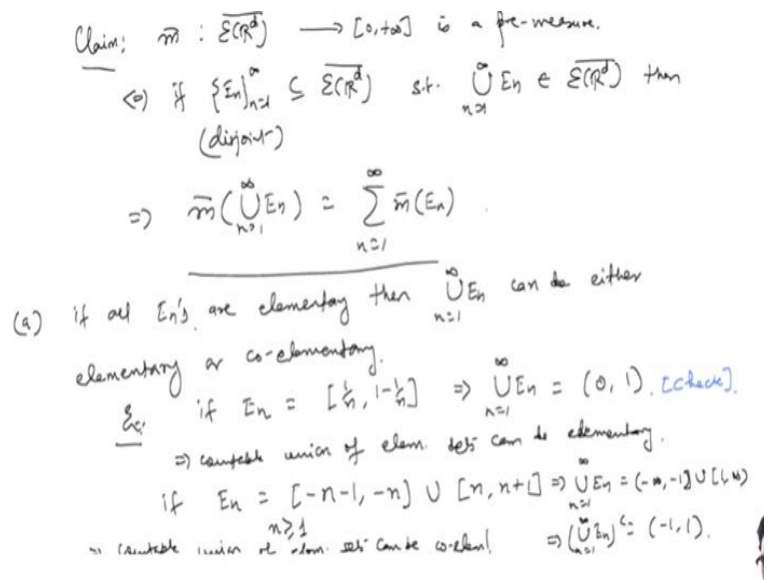

On the other hand before I prove that notice that union of co-elementary sets can never be elementary. Because each of them is unbounded and the union is bounded so which is a contradiction. So union of co-elementary sets can never be elementary but the union of co-elementary sets can be co-elementary. So in this case for example if you take En to be minus infinity to –n union n to infinity then En’s are co-elementary and the union of En’s n = 1 to infinity is again co-elementary

Because if you write this as E as E complement is nothing the set -1 the interval -1 to 1 so it is in the complement is elementary so it is co-elementary. So we see that union of co-elementary sets can never be elementary but it can be co-elementary. So for the second case in this case if En is a disjoint collection of co-elementary sets which implies that the union n = 1 to infinity is always co-elementary is a co-elementary set.

So this is means that the measure of the union is plus infinity on the other hand this sum of this measures is also plus infinity because each term is plus infinity so in fact it shows that even if one of them is co-elementary then you have plus infinity on both sides and so the equation holds. So even if 1 En is co-elementary so let me write not just disjoint collection of co-elementary sets but disjoint collection of elementary and at least 1 co-elementary set.

Then this means that the union is a co-elementary set and you have plus infinity on both sides this implies that countable additivity holds. So now we only have to show case a in when where union of elementary sets is elementary and union of elementary sets is co-elementary so let us see the proof.

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Let $\{E_n\}_{n\geq 1}^{\infty}$ be elementary (disjoint) sets s.t. $\bigcup_{n\geq 1}^{\infty} E_n$ is elementary

To show: $m\left(\bigcup_{n\geq 1}^{\infty} E_n\right) = \sum_{n\geq 1}^{\infty} m(E_n)$

Since $\bigcup_{n=1}^{N} E_n \subseteq \bigcup_{n=1}^{\infty} E_n$

$\Rightarrow \sum_{n=1}^{N} m(E_n) \leq m\left(\bigcup_{n\geq 1}^{\infty} E_n\right)$ for any $N \geq 1$

$\Rightarrow \lim_{N\to\infty} \sum_{n=1}^{N} m(E_n) \leq m\left(\bigcup_{n\geq 1}^{\infty} E_n\right)$

$\Rightarrow \sum_{n\geq 1}^{\infty} m(E_n) \leq m\left(\bigcup_{n\geq 1}^{\infty} E_n\right)$

So let En be elementary sets such that elementary disjoint sets such that the union is again elementary. So then we have to show that m of the union because it is an elementary so m bar is just m is the sum of the m En’s is n = 1 to infinity. So here we do not have countable sub-additivity because we do not know that m is an outer measure. So we have to show countable sub additivity on the other hand it is easy to show that since any finite union n =1 to N En is the subset of the En's.

This implies that by finite additivity that this finite sum of m En is less than or equal to the measure of the union and so if you this is for n En greater than equal to 1. Here I have used the fact that the elementary measure is finitely additive on disjoint sets. So here take the limit as S tends to infinity N tends to infinity n = 1 to N En is less than or equal to m of the union. So this is nothing but the limit is nothing but the whole sum so we get the inequality on one side.

The other inequality is if N was an outer measure we would then conclude that the equality holds. But we do not have countable sub-additivity so we have to improvise so we have to give another argument.

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it suffices to show: $m(\underbrace{\bigcup_{n=1}^{\infty} E_n}_{E}) \leq \sum_{n=1}^{\infty} m(E_n)$. ← outer approx.

inner approx. → E

we will use the elementary measure has the property of inner approx. by closed elementary sets & outer approx. by open elementary sets.

Given $\epsilon > 0$, $\exists$ a closed elementary set $F \subseteq E$ s.t.

$m(E \setminus F) \leq \epsilon$ (F is a compact set).

$\Rightarrow m(F) \geq m(E) - \epsilon$.

For each $n \geq 1$, $\exists$ an open elementary set $U_n \supseteq E_n$ s.t. $m(U_n) \leq m(E_n) + \frac{\epsilon}{2^n}$.

$\Rightarrow F \subseteq \bigcup_{n=1}^{\infty} U_n$

$\Rightarrow \exists$ a finite collection $\{U_1, U_2, \dots, U_N\}$ s.t. $F \subseteq \bigcup_{n=1}^{N} U_n$

So it suffices now it suffices to show that m of the union is less than or equal to the infinity sum of m En's. So to do this we proceed as follows so let me denote this set by E we will use the fact that the elementary measure as the property of inner approximation by closed sets closed elementary sets in fact and outer approximation by open elementary sets. So for E we will use inner approximation and for En's we will use outer approximation and then we will use a compactness arguments to reduce to finitely many unions and then we can use finite additivity.

So let us see how this works so given epsilon greater than 0 there exists closed set F inside E closed elementary sets F inside E such that they elementary measure of E –F is less than or equal to epsilon. And because F is a closed elementary set it is also bounded and close set which means that F is compact set. So this means that the measure of F is greater than or equal to measure of E – Epsilon.

On the other hand their exist for each n greater than equal to 1 their exist an open set Un containing En again open elementary set we are only working with elementary sets. So an open elementary set Un containing En such that the measure of Un is less than or equal to measure of En + epsilon by 2 to the power n. So again we are going to use it 2 to the power epsilon by 2 to the power (()) (21:21) because there are sum involved.

So then this implies that F is contained in the union n = 1 to infinity of Un because F is the subset of the unions of En’s. And since F is compact their exist a finite set finite collection say U1 up to renumbering I can say it is U1, U2 up to UN say such that F is contained in the finite collection of a sets Un.

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$$\Rightarrow \quad m(F) \leq m\Big(\bigcup_{n=1}^{N} U_n\Big) \underbrace{\leq}_{\text{Finite subadditivity}} \sum_{n=1}^{N}\Big(m(E_n) + \frac{\epsilon}{2^n}\Big)$$

$$\leq \sum_{n=1}^{\infty}\Big(m(E_n) + \frac{\epsilon}{2^n}\Big) = \sum_{n=1}^{\infty} m(E_n) + \epsilon.$$

$$\Rightarrow \quad m(E) \leq \sum_{n=1}^{\infty} m(E_n) + 2\epsilon$$

$$\Rightarrow \quad m(E) \leq \sum_{n=1}^{\infty} m(E_n).$$

Now suppose that $\{E_n\}_{n=1}^{\infty}$ is a collection of elementary sets s.t. $E = \bigcup_{n=1}^{\infty} E_n$ is co-elementary. $\Rightarrow m(E) = +\infty$.

So it suffices to show that $\sum_{n=1}^{\infty} m(E_n) = +\infty$. $\Leftrightarrow$ Given any $M > 0$ $\exists N \in \mathbb{N}$ s.t. $\sum_{n=1}^{N} m(E_n) \geq 1$

So this means that the measure of F is less than or equal to the measure of the union n = 1 to N Un but this is less than or equal to the sum n =1 to infinity 1 to n first N measure of En + epsilon by 2 to the power n and this is less than or equal to note that here I used finite sub additivity property for this inequality finite sub-additivity property which follows from finite additivity actually. So then we have this finite sum is of course less than or equal to the infinite sum mEn + epsilon / 2 to the power n which is nothing but the total sum m En + epsilon.

And on the other hand this implies that m E – epsilon is less than equal to m = 1 to infinity m E and + epsilon. So this minus epsilon can be taken on the right side so you get + 2 epsilon nevertheless you get the inequality that we want n = 1 to infinity m En. So we have shown this inequality and we have proved in fact that for disjoint collection of elementary sets which are whose union is again elementary you have the premeasure condition is valid.

Now suppose that the union so all En's is a collection of elementary sets such that the union E n = 1 to infinity is co-elementary. So this means that the extended measure so these are all so the extended measure is plus infinity so it suffices to show that the sum of m En n = 1 to infinity is also equal to plus infinity which is to say that given any N greater than 0 their exist N is the natural numbers such that this sum the partial sum from n =1 to N m En is greater than or equal to N.

So this shows that when you take the limit as N goes to infinity you get plus infinity so we proceed as follows.

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Since $\bigcup_{n=1}^{\infty} E_n$ is co-elementary $\Rightarrow$ $\bigcap_{n=1}^{\infty} E_n^c$ is elementary

So we can choose an open elementary set $U \supseteq \bigcap_{n=1}^{\infty} E_n^c$ (i.e. $m(U) < \infty$).

$$\Rightarrow \bigcup_{n=1}^{\infty} E_n \supseteq U^c$$

For any $r > 0$ which is large enough so that the ball $\overline{B(0,r)}$ contains U:

$$\bigcup_{n=1}^{\infty} (E_n \cap B(0,r)) \supseteq \underbrace{\underbrace{\overline{B(0,r)}}_{\text{closed \& bdd}} \cap \underbrace{U^c}_{\text{closed}}}_{\text{a compact set } K.} = \overline{B(0,r)} \setminus U$$

$$\Rightarrow K \subseteq \bigcup_{n \in I_r} E_n \quad ; \; I_r := \{ n \in \mathbb{N} : E_n \cap \overline{B(0,r)} \neq$$

Since the union of En's n = 1 to infinity is co-elementary this implies that the intersection of the complements n = 1 to infinity is elementary. So we can choose an open elementary set U such that it contains this intersection which is elementary of the complements of En’s. So in particular we have the elementary measure is finite so now this means that once you again take complements this means that the union of En’s is now contained inside the complement of this open set U.

Now for any r greater than 0 which is large enough to contain U is a bounded set so we can always find for any r which is large enough. So that the ball the Euclidian ball with center 0 and radius r contains U for this r we have the intersection E and intersection B 0, r. I would rather take the closed unit ball closed ball with radius r here and you will see y. So this is contained in the intersection of complement of u with this closed ball of radius r.

And this is nothing but B 0, r – u and so this is in fact a closed and bounded set u is open so u complement is closed and this is also closed and bounded. So this whole thing is a compact set k. So therefore this implies that there is a slight mistake here this should be inclusion should be the other way round. So here we have a compact set k which is contained inside this infinite union but in fact we can discard those sets En which do not intersect B 0, r.

So this compact set k is contained in the union of En’s belong where n belongs to some index set Ir which is given by the set of natural number such that En intersection B 0, r bar is not empty. Because the others if they are empty we can discard them so now if Ir as finite cardinality then we can use the finite additivity property to conclude.

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if $|I_r| < +\infty$

$$\Rightarrow \quad m(k) \leq m\Big(\bigcup_{n \in I_r} E_n\Big) = \sum_{n \in I_r} m(E_n)$$

$$\leq \sum_{n=1}^{\infty} m(E_n). \quad \text{(holds for any } r>0 \text{ large enough).}$$

$$m(K) = m(\overline{B(0,r)}) - m(u). = C(d) \cdot r^d - m(u)$$

So, as $r \to \infty$ $\quad m(k) \to \infty.$

$$\Rightarrow \quad \sum_{n=1}^{\infty} m(E_n) = +\infty.$$

if $|I_r| = +\infty$. Given $\epsilon > 0$, we chose for each $n \geq 1$ an open elementary set $U_n \supseteq E_n$ s.t. $m(U_n) \leq m(E_n) + \frac{\epsilon}{2^n}$.

then $k \subseteq \bigcup_{n=1}^{\infty} U_n \Rightarrow \exists$ a finite collection of U_i's s.t. $k \subseteq \bigcup^{N} U_n$.

Because this implies that measure of so measure of k is less than or equal to so first if the cardinality is finite then this is less than or equal to the measure of a finite union of disjoint sets which is then equals to this sum over a finite index set by finite additivity. And then you can take the entire sum on the right hand side and note that m K = m B 0, r bar – m u and this is equal to some constant C which only depends on the dimension d times the radius to the power d minus of fixed constant m u.

So as r goes to infinity m k goes to infinity and because we proved this for any arbitrary r this inequality holds for any r large enough. So this means that the sum is plus infinity because the left hand side becomes bigger and bigger and the right hand side gets pushed to plus infinity. But if it could happen that even with the intersection of small of a bounded Euclidian ball there could be infinity many elementary sets inside this collection En that intersect this ball.

So that the cardinality of Ir may be plus infinity so, in that case we choose so given epsilon greater than 0. We choose for each n greater than equal to 1 and open elementary set un containing En such that the measure of un is less than equal to the measure of En + epsilon by 2 to the power n. So we are again going to use this compactness strict that we use before in the previous case to conclude here.

Because then their exist then k is contained in the union of all this Un's and this means that their exist a finite collection of Un's such that k is the subset of this finite collection 1 to N of Un.

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$$\Rightarrow \quad m(K) \leq m\left(\bigcup_{n=1}^{N} U_n\right) \leq \sum_{n=1}^{N} m(U_n)$$

$$\leq \sum_{n=1}^{N}\left(m(E_n) + \frac{\epsilon}{2^n}\right) \leq \sum_{n=1}^{\infty}\left(m(E_n)\right) + \epsilon.$$

Chose $\epsilon = 1$.

$$\Rightarrow \quad \sum_{n=1}^{\infty} m(E_n) \geq m(K) - 1$$

and $m(K) \to \infty$ as $r \to \infty$.

$$\Rightarrow \quad \sum_{n=1}^{\infty} m(E_n) = +\infty.$$

$$\Rightarrow \quad \bar{m} : \overline{\mathcal{E}(\mathbb{R}^d)} \to [0, +\infty] \text{ is a pre-measure.}$$

So then again we can use finites sub-additivity property so m k is less than or equal to the measure just as that we did before this is less than or equal to the finite sum mu N which is then less than or equal to the sum n = 1 to N m En + epsilon by 2 to the power n which is again less than or equal to n = 1 to infinity measure of En + epsilon. So epsilon is a fixed quantity so it remains finite.

So for example one can choose epsilon to be 1 in which case this sum n = 1 to infinity m En is greater than or equal to mK – 1 and mK goes to infinity as r goes to infinity. So again we get that this sum is bounded above by a sequence of numbers which goes to infinity. So this again means that n = 1 to infinity m En is plus infinity so we see that we have finally proved that this proves that the elementary measure m bar on the elementary algebra is a premeasure.

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Hahn-Kolmogorov extension thm $\rightsquigarrow$ a measure $\tilde{m}$ on a σ-alg. $\mathcal{C}_{m^*}(\mathbb{R}^d)$,

where $m^*(E) := \inf\left\{ \sum_{n=1}^{\infty} m(E_i) \; : \; E \subseteq \bigcup_{n=1}^{\infty} E_i \, , \; E_i \in \overline{\mathcal{E}(\mathbb{R}^d)} \text{ for each } i \right\}$

(Ex).

$$= \inf\left\{ \sum_{n=1}^{\infty} m(B_i) : E \subseteq \bigcup_{n=1}^{\infty} B_i , \; B_i \text{ boxes} \right\}.$$

So the elementary measure gives the Lebesgue measure on $\mathbb{R}^d$ via the Hahn-Kolmogorov extension thm.

And once we have a premeasure then we can use the Hohn Kolmogorov extension theorem to get a measure extension theorem gives a measure and tilde on a sigma algebra which is given by C m star of Rd where m star of E is the outer measure defined using a elementary sets. So this is the sum n = 1 to infinity m Ei such that E is the subset of Ei and Ei belongs to this elementary algebra for each i. Of course if one of these is co-elementary then we can discard this so this is an infimum over such things.

So we can discard those covers because then you will get plus infinity on the sum so in fact one can show that this is the same as the outer measure that you get from just using boxes Bi boxes. So this is an exercise for you to check that these 2 things are the same therefore the Lebesgne outer measure and the measure induced by the elementary measure which is the premeasure is they are the same and therefore the final measure is just the Lebesgne measure.

So the elementary measure is gives the Lebesgne measure on Rd via the Hohn Kolmogorov extension theorem