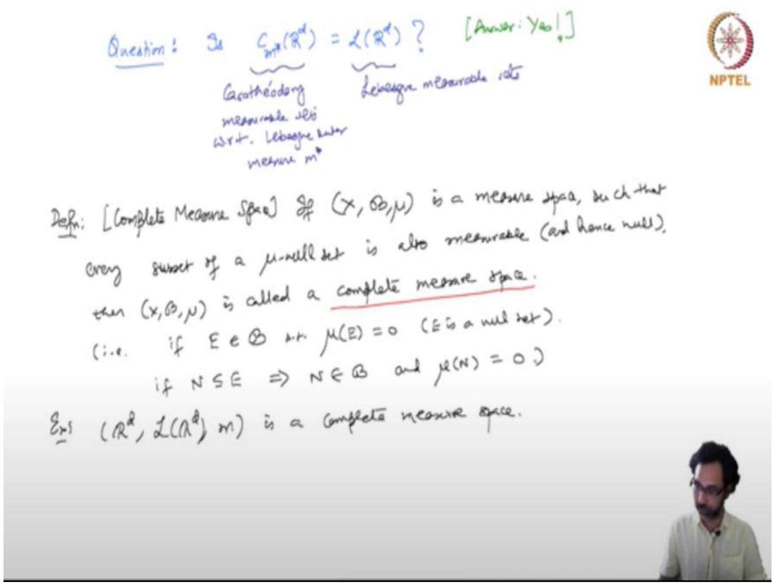
Measure Theory

Prof. Indrava Roy
Department of Mathematics
Institute of Mathematical Science

Module No # 08
Lecture No # 36
Lebesgue measurable class versus Caratheodory extension of usual outer measure on Rd

(Refer Slide Time 00:16)

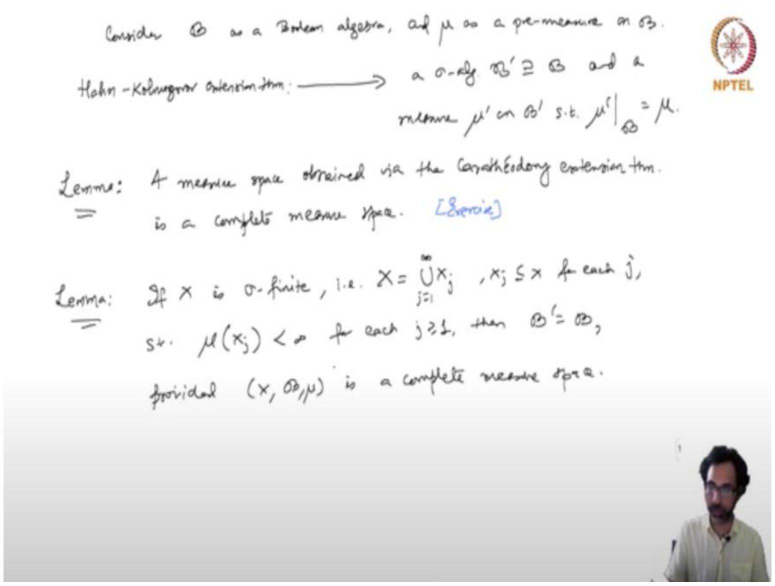


So now we are in the position to answer the following question is Cm star Rd = L Rd meaning this the left hand side is the collection of caratheodory measurable sets with respect to the Lebesgue outer measure m star and on the right hand side we have the collection of Lebesgue measurable sets as we have defined them before. So the question rises whether if you use the Lebesgue outer measure m star and define the caratheodory measurability condition and get a sigma algebra this sigma algebra whether it is the same as the Lebesgue sigma algebra or not.

And they answer to this question is yes answer is yes. And we will prove it shortly using some abstract nonsense if you want. So for this let me make a quick definition this is of a complete measure space. So if X, B Mu is measure space such that every subset of a Mu null set is also measurable and hence null then X, B Mu is called a complete measure space. So this is to say that if E is an element of the sigma algebra B such that mu E is 0. So E is a null set and if N is a subset of E then this should imply that N belong to B and Mu N is also equal to 0.

So this is the condition that should be satisfied for a measure space to be a called a complete measure space. So of course an example we already know is the Lebesgue collection of Lebesgue measurable sets this is with the of course the Lebesgue measure so Rd L Rd and M the lebesgue measure m is a complete measure space. Because when you have a lebesgue outer measure 0 then a set becomes measurable. So the prototypical example of a measure space given by the Lebesgue outer measure on Rd Lebesgue measure on Rd is a complete measure space.

(Refer Slide Time 04:17)

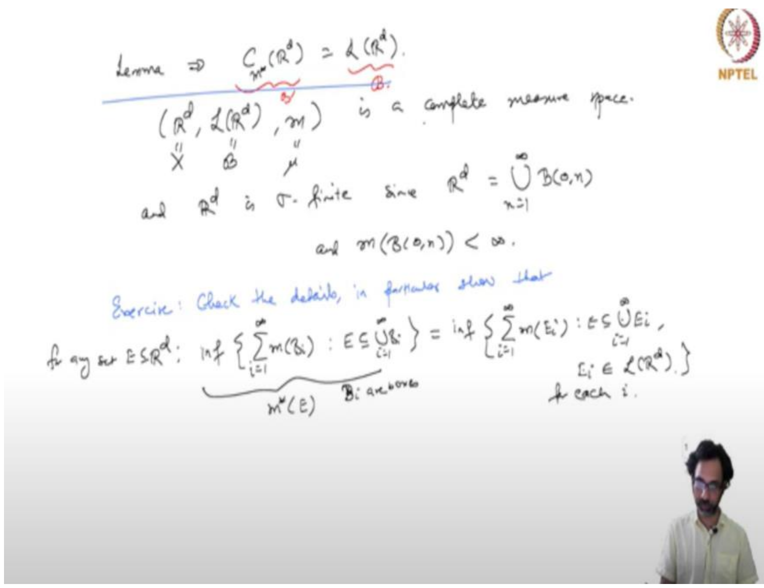


Now consider B the sigma algebra B has a Boolean algebra and Mu has a premeasure. So of course, a Boolean algebra is sigma algebra is a Boolean algebra and a pre measure, a measure is a pre measure. So let me repeat sigma algebra is a Boolean algebra and a measure is a premeasure. So you can consider B as a Boolean algebra and Mu as a pre measure on B ok. Then the Hahn Kolmogorov extension theorem gives another sigma algebra a sigma algebra B prime containing B and a measure Mu prime on B prime such that Mu prime restricted to B equals Mu.

So we are using Hahn Kolmogorov extension theorem for the specific case when you already have a measure and sigma algebra B. Then we can generate another sigma algebra B prime and a measure Mu prime such that Mu prime equals Mu. Now note that anything that you get from the caratheodory extension theorem is going to be a complete measure space. So I will put this as a lemma to check for you. So a measure space obtained via the caratheodory measurability condition or the caratheodory extension theorem is a complete measure space.

So this you can check now the question arises whether B prime is equal to B or not. And this is answered in the following lemma. So this is left as an exercise for you. And this lemma says that if x is sigma finite meaning that x is a union of countability many subset x j is a subset of each j. Such that Mu of xj is finite or each j then you can say that B prime is equal to B. So when you already have a complete measure space on a sigma finite space then you have B prime is going to be equal to B. So here we are assuming so provided that x B Mu is a complete measure space.

(Refer Slide Time 08:24)



So let us see why this lemma imply that I claim that this lemma implies that C m star Rd equals L Rd because when you take the start with the measure space Rd L Rd so this is our x this is our B and this is our Mu. This is our complete measure space and then we note that an Rd is sigma finite since Rd can be written as the union of Euclidean balls with center 0 and radius n. And of course the Lebesgue measure of these balls B 0 n is finite.

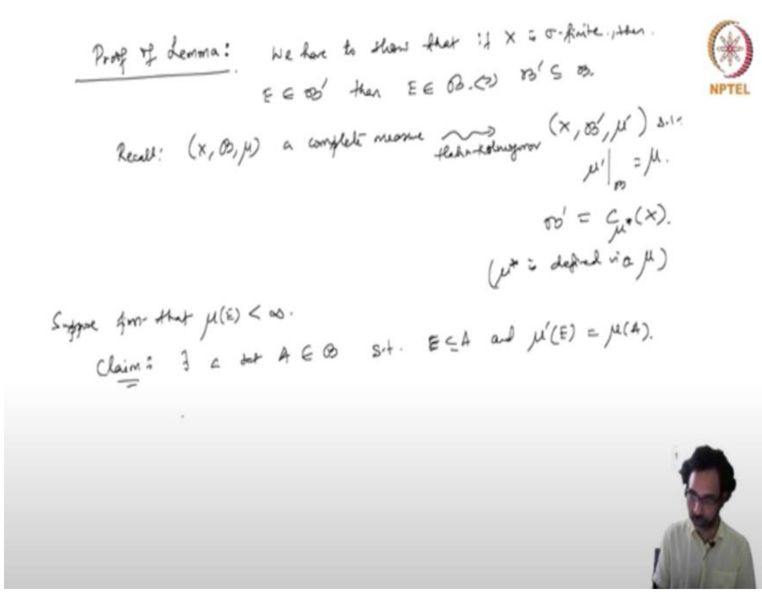
So Rd sigma finite with respect to the Lebesgue measure and so both these condition for our lemma are satisfied which is that x be a sigma finite and that it should be a complete measure space. So this implies that this B prime when so B prime is nothing but this C m Rd this is B prime and this is B. So the lemma says exactly that B prime is equal to B. So we would have shown that once we show the lemma we would have shown that the caratheodory collection of caratheodory measurable sets is equal to the collection of Lebesgue measurable sets.

So I leave it to you as an exercise to check the details of this implication that the lemma implies this check the details in particular show that we have the infimum of so for any set E in Rd the infimum of the elementary measures of boxes such that E is covered by the union of the boxes is equal to so this is nothing but of course the Lebesgue outer measure E. But this is also equal to the infimum of the sums when you replace boxes by measurable sets rather than. So you replace boxes by measurable sets in general.

So here Bi are boxes and here Ei belong to the Lebesgue sigma algebra for each i. So one has to show this equality to conclude that because the left hand side is the Lebesgue outer measure and the right hand side is the outer measure. So here I should rather write now it is ok. So this is the Hahn Kolmogorov prescription when you go from pre measure to an outer measure and then use the caratheodory measurability.

So we have to show that the 2 outer measures given by these 2 formulas are exactly the same. So once you show this then the lemma implies that the collection of caratheodory measurable subset of Rd with respect to the lebesgue outer measure is the same as the collection of Lebesgue measurable subset of Rd.

(Refer Slide Time 13:04)



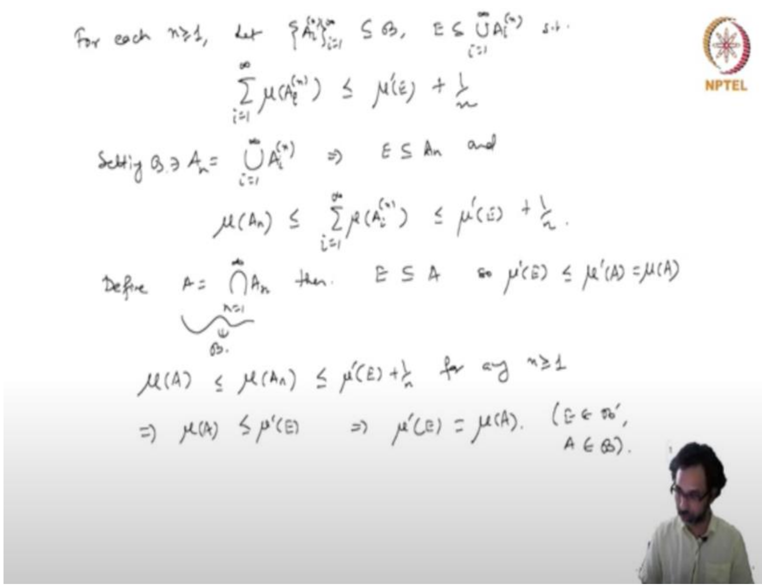
So now let us see the proof of this lemma. So we have to show that if E belongs to B prime then E belongs to B. So remember that B prime so remember that we had X B Mu a complete measure space which gave via the Hahn Kolmogorov extension theorem. Another sigma algebra

B prime and Mu prime on B prime such that Mu prime when restricted to B is exactly Mu. And so we can write B prime here as C Mu of well not Mu but Mu star which is generated the outer measure induced by the pre measure Mu. So this Mu star is defined via Mu.

It is a premeasure defined via Mu this is the Hahn Kolmogorov procedure to go from Boolean algebra and a premeasure to a complete measure space. Now we have already starting with the complete measure space and we have to show that whenever we have x is sigma finite then B prime is a sub collection of B. And so B prime is equal to B. So how do we show this? So, if suppose first that the measure of E is finite.

Then I will claim that there exist a set A in B such that E is a subset of A and Mu prime of E is equal to Mu of A. So this is for any set E in the bigger sigma algebra B prime. So let see how to prove this.

(Refer Slide Time 15:59)

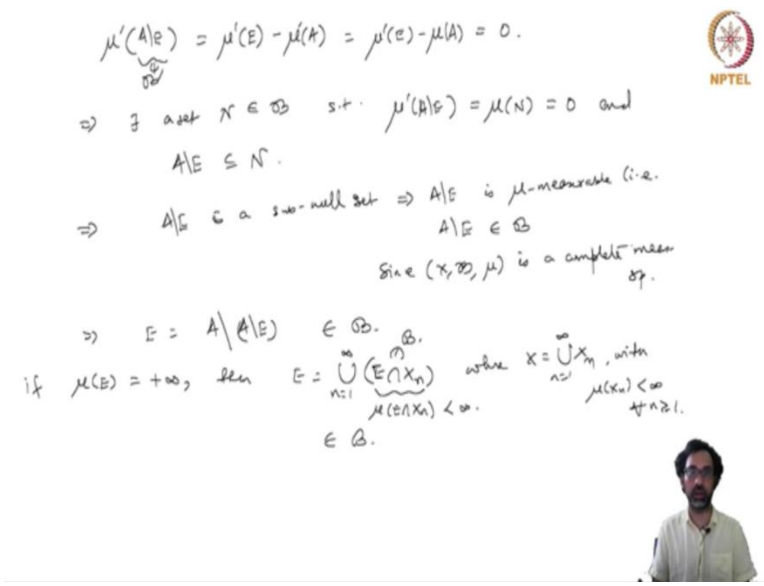


So to prove this claim to proceed as follows so for each n greater than or equal to 1 let Ai be a collection of elements in B covering E. So Ai n i = 1 to infinity such that the sum mu Ai n i = 1 to infinity less than or equal to Mu prime E + 1 over n. So set An to be the union of Ai n's which means that E is a subset of An and Mu An. So this belongs to B so we can write Mu An is bounded above by this sum Mu Ai n. And then this bounded by Mu prime E + 1 over n.

So now if we define A to the intersection of all these An is n = 1 to infinity then E is a subset of A. So Mu prime E is less than or equal to Mu prime A which is equal to Mu A because again this A belong to B because it is sigma algebra So we have Mu prime E less than or equal to mu A. On the other hand Mu A is less than or equal to Mu A n for m E n and so this is less than or equal to Mu prime E + 1 over n for m E n greater than or equal to 1 this means that Mu A is less than or equal to Mu prime E.

And this means that mu prime E is equal to Mu A. So we have shown that for any set E in B prime. So E was in B prime and there exists a set A in B such that Mu prime E equal MA.

(Refer Slide Time 18:25)



So now if you take the measure Mu prime of E - A this is equal to Mu prime E - Mu A right. Because well it is equal to Mu prime E - Mu prime A, and this is equal to Mu prime E - Mu A and this is equal to 0. As we have shown here I am using that Mu prime itself is also a measure. So we have this so now using again the fact that this belongs to B prime. And so this implies that there exists a set N in B such that mu prime E - A is equal to Mu N = 0.

And E - A is a subset of this set N. From what we have just shown here again we have a set of finite measure in B prime. So we can find a measure find the set An in the sigma algebra B for which the measure agree and in this case it is 0. So this mean that E - A is a sub null set which means that E - A is Mu measurable this is E - A belongs to B because x B Mu is a complete measure space.

So E - A belongs to B. So here actually I should rather take A - E rather than E - A because E is a subset of A. So we have A - E everywhere rather than E - A. So now E also here A - E so E can be written as A - A - E and so this belongs to B. So this proves the statement when E has finite measure. So if Mu E is infinite then E can be written as countable union n = 1 to infinity of the intersection with X n.

Where x is the union of xn n = 1 to infinity with Mu x n finite for all any. So now we can say that even if E has infinite measure then each of these is finite measure mu of E intersection x n is finite. And so therefore this each of these sets belongs to B and so the union belong to B and then we have done. So we showed that any completion that you get for a complete measure space via the caratheodory extension theorem gives you nothing extra and you get the same sigma as you brought back when X is sigma finite.