Measure Theory
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Module No # 07
Lecture No # 35
Abstract measure and Hahn-Kolmogorov Extension

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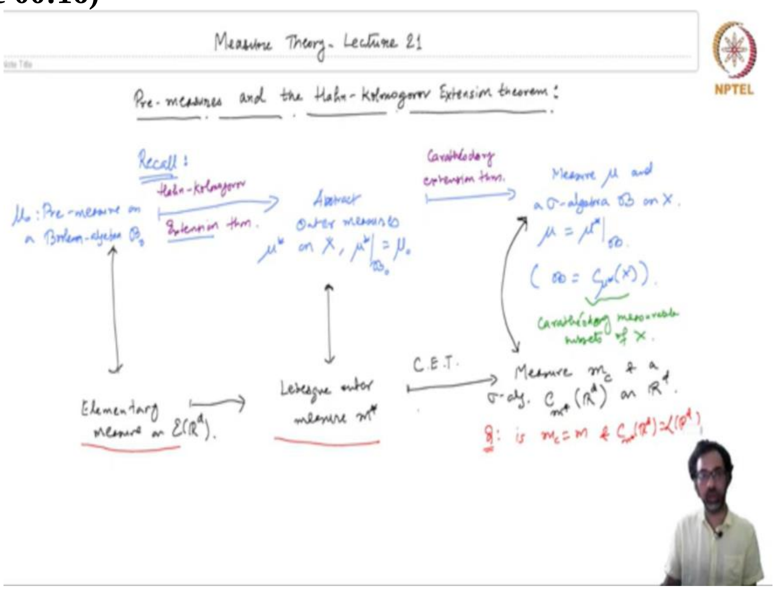

The topic of this lecture is premeasures and Hahn Kolmogorov extension theorem. So let me briefly recall what we have done and we will put this new topic todays topic in that context as well. So recall that we have defined what are called abstract outer measures and from abstract outer measures we have given a theorem which is the caratheodory extension theorem. Which once you have an abstract outer measure it the caratheodory extension theorem via the notion of the caratheodory measurability.

This gives you a measure a sigma algebra under countability additive measure on that sigma algebra. So let me write here abstract outer measure on Mu star on x and this gives you the caratheodory extension theorem gives you a measure Mu and sigma algebra B on x. And here we have Mu is equal to simply the restriction of the outer measure to this sigma algebra B. So we denoted our B was C Mu star of x which was the collection of caratheodory measurable subset is collection was caratheodory measurable subsets of x.

So this was the restriction of the caratheodory extension theorem once you have an outer measure how to produce sigma algebra and measure which satisfies not only countable Sub additivity but also countable additivity for disjoint sets in the sigma algebra. Ok so here of course there is a parallel with what we have seen before so the abstract outer measures are exactly generalization of the Lebesgue outer measure and so the caratheodory extension theorem gives you measure.

So let me write it as m star and the caratheodory extension theorem let me write it C.E.T for short it gives you measure mc c for caratheodory and a sigma algebra C m star of Rd on Rd. So now the question arises whether this measure in mc is precisely our lebesgue measure is mc = m Lebesgue measure and C m star Rd = L Rd. So the answer to this question is affirmative and this gives you another way to define Lebesgue measure as the measure induced by the caratheodory extension theorem the Lebesgue measure is the measure induced by the caratheodory extension theorem from the Lebesgue outer measure.

And the sigma algebra that it gives is exactly the sigma algebra of Lebesgue measurable sets on Rd and the measure is exactly the Lebesgue measure. So we will answer this question here but this specific instance of Lebesgue measure is when a generalization for the sorry so this measure mu that we get and the sigma algebra C mu star x on an arbitrary star x is then a generalization of the Lebesgue measure that we have already seen.

On the other hand we could remember that our abstract our Lebesgue outer measure came from something more basic which was the elementary measure of a Jordon measure. In fact we define the Lebesgue outer measure using only boxes and so it was given by the elementary measure on the elementary subset of Rd.

So then the question arises whether you can take abstract version of what is an elementary measure such that it will give you this abstract outer measure mu star on X. So this concept is precisely what is called pre measure on a Boolean algebra B naught. And this passes from pre measure to abstract outer measure is exactly this a Hahn Kolmogorov extension theorem. So this bottom row which goes from the elementary measure to the Lebesgue outer measure and then to

Lebesgue measure provided we prove that we get exactly the Lebesgue measure using the caratheodory condition as well.

Then is generalized to this concept of premeasure and the Boolean algebra which; then gives you the abstract of outer measure. Which; then gives you a measure on this on the sigma algebra on the sigma algebra of caratheodory measurable subsets. And in fact this measure mu star this outer measure Mu star given by the Hahn Kolmogorov extension theorem also satisfies that the restriction of this outer measure on this Boolean algebra B naught on which we started is precisely this Mu naught which is the pre measure.

So let me write it here Mu naught pre measure on a Boolean algebra B naught and this gives you an abstract outer measure which when restricted to sets B naught gives you Mu naught. Which is why it is called as a extension theorem which because it is extending this pre measure to outer measure. And then the further extension theorem due to caratheodory then gives you that the measure on the full sigma algebra B or C Mu star X

So this is the layout of our conceptual layout of our plan which is to either start from an abstract outer measure. If you are given a abstract outer measure then you can use the caratheodory extension theorem to go to a measure. But if you are already given a premeasure then you can get an outer measure and then a measure using the Hahn Kolmogorov extension theorem. So we will also answer for in this course that even if you do not have a pre measure you can get an outer measure given a very mild condition.

So abstract outer measures are quite easy to achieve and then we will also answer that for the specific case of Euclidean space Rd whether the measure induced by caratheodory extension theorem from the lebesgue outer measure is precisely the lebesgue measure or not. So this is our plan for today's lecture. So let me begin by defining what is a pre measure?

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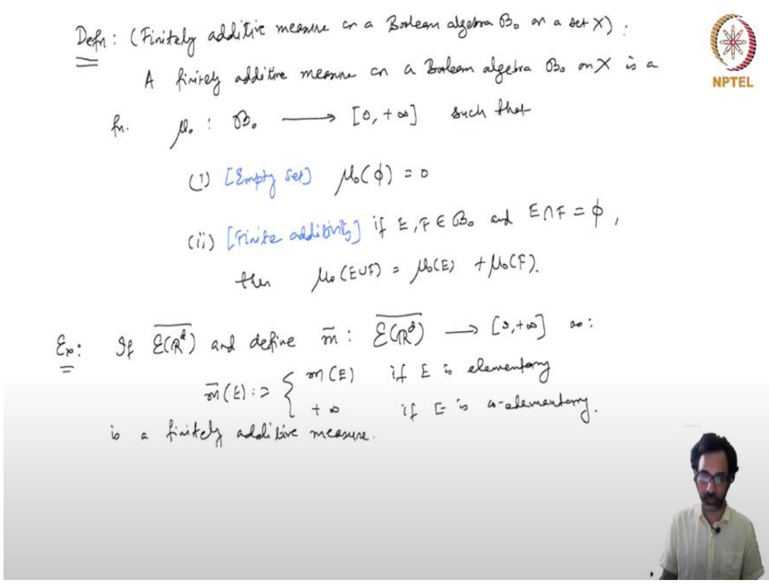

First before defining what is a premeasure? We define what is called finitely additive measure on a Boolean algebra B naught on set x. So on a finitely additive measure on Boolean algebra B naught on x is a function Mu naught which takes elements; of B naught. So the subset of x lying in the Boolean algebra and gives you a non-negative extended real number. So it can lie between 0 and infinity with both end points included and, such that it satisfies two axioms.

So the first one is the empty set axioms, which is that mu naught of the empty set should be 0 and the second one is finite additivity axioms. The finite additivity axioms says that if E and F belong to B naught and they are disjoint E intersection F is empty. Then mu naught of E union f = Mu naught E + Mu naught F. So for example if you take the elementary algebra E Rd and define m from this elementary algebra E Rd bar 2, 0 + infinity here I am using the annotation m.

So it should not come as surprise it is going as an elementary measure which is defined as m E = the elementary measure of E well I should may be change m to m bar. Here m bar E is the elementary measure of E if E is elementary and equals plus infinity if E is co elementary. So we call it as the elementary algebra was the collection of elementary and co elementary subsets of Rd. So for the elementary subset we have the elementary measure.

And for the co elementary subsets we have the value plus infinity. So this is a finitely additive measure. As one can check quite easy to check because we have already proved that it is finitely additive on elementary set subset disjoint elementary subsets. But if one of them is co elementary then both sides are plus infinity end basically you have done. So this is a finitely additive measure.

Similarly if you take the Jordan measure algebra and you can similarly define as extended Jordon measure for Jordan and co Jordan measurable subset and you will get again finite subsets. So this is a prototypical example that we will be working with.

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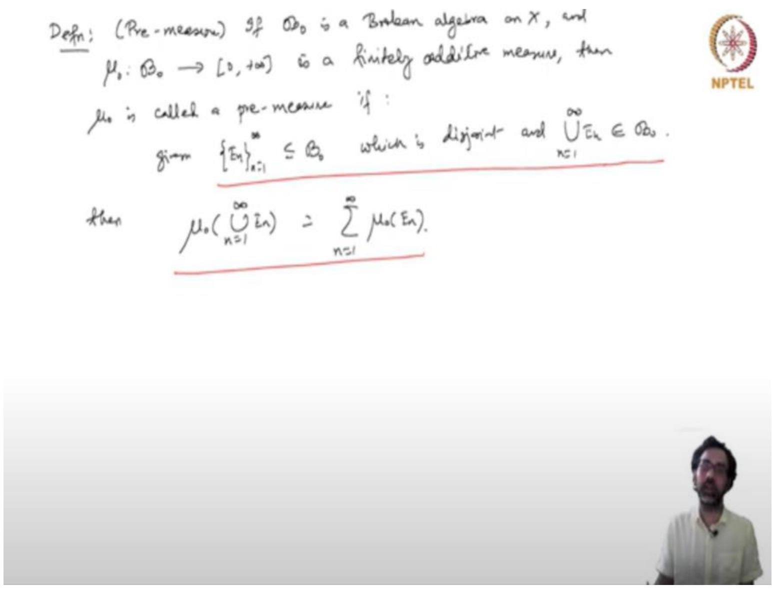

So, now a premeasure is defined as follows so if B naught is a Boolean algebra on x and Mu naught is a pre measure is a finitely additive measure then Mu naught is called a pre measure if it satisfies one further property. If given a collection m = 120 En of elements of B naught which is disjoint collection of sets within B naught. And the union m = 120 infinity also lies in B naught then the Mu naught of this union countable union now is equal to the sum m = 1 infinity Mu naught En.

So of course if you want to extend a pre measure to a measure then they should be a necessary condition this countable additivity on the Boolean algebra itself should be a necessary condition meaning exactly this part that I wrote here. That if you have a disjoint collection such that the union also belongs to the Boolean algebra then you should have the countable additivity property satisfied by a pre measure so that it can become a measure. So if you expecting something to become a measure then it should already satisfy this countable additivity property for unions which again lie in the Boolean algebra.

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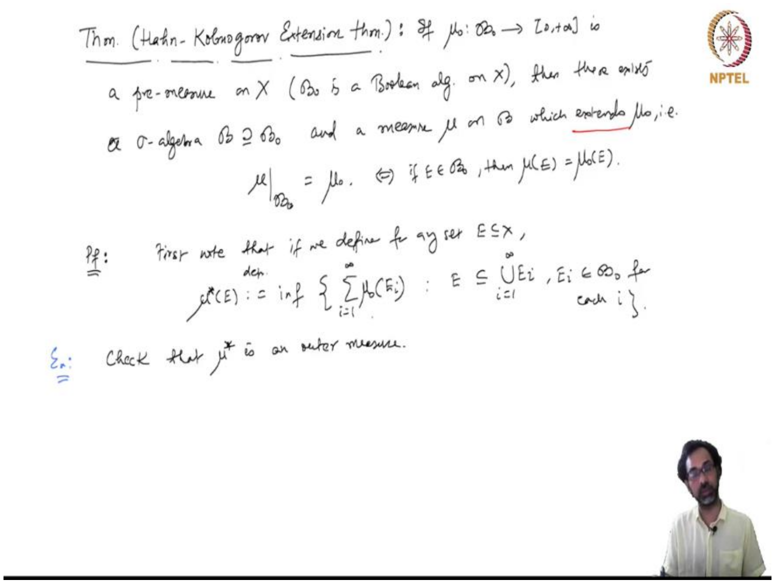

So now we are ready to state the Hahn Kolmogorov extension theorem and this says that if Mu naught is a premeasure on X. So here B naught is a Boolean algebra on X. Then there exist a sigma algebra B naught B containing B naught and measure Mu on B naught B sorry measure Mu on B which extends Mu naught. Which means that when you restrict Mu to elements of B naught you get Mu naught or in other words if E is the elements of B naught then Mu of E = Mu naught of E.

So this is the extension part in the Hahn Kolmogorov extension theorem because we are extending a premeasure to a measure on full sigma algebra. So let see the proof for this wonder theorem. First note that if we define for any set E in X Mu star E to be so this is the definition we define it to be the infimum of the sums i = 1 to infinity m Ei such that E can be covered by this collection of sets i = 1 to infinity Ei. And each Ei belongs to the Boolean our algebra B naught for each i.

So as you can see this is almost replica for the definition of the Lebesgue outer measure so here I should rather write Mu naught because we do not have the elementary measure but rather the premeasure. So this my mistake says that of course we just repeating the definition of the lebesgue outer measure which was define using coverings of sets by countably boxes and then taking the infimum of the sums 1 to infinity of this of each elementary box.

So here we are replacing the boxes by elements of our Boolean algebra B naught which is given. And then we are doing exactly the same thing as before. So we can immediately check that Mu star is in fact an outer measure. So this I will leave you to do this as an exercise. And in fact we need not even have this premeasure to define the outer measure. In fact if you go to the proof of the definition of the Lebesgue outer measure and the proof that it satisfies all the axioms of outer measure.

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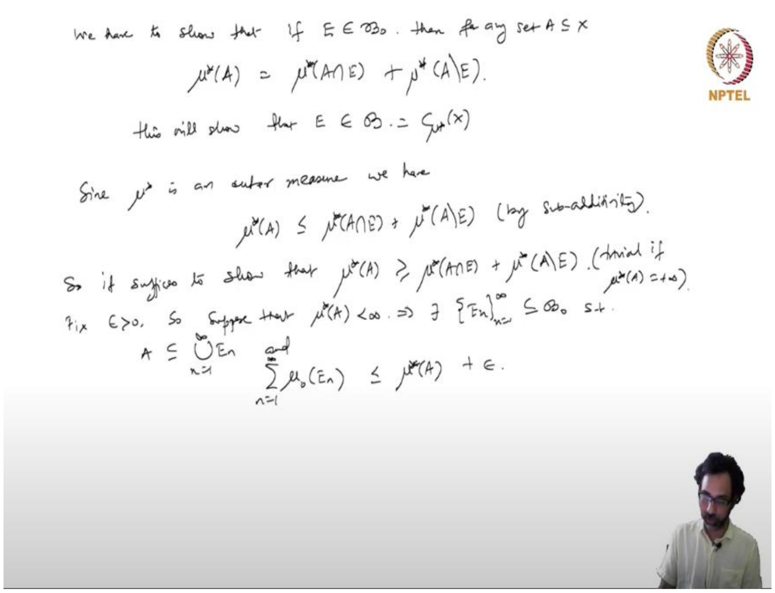

Then you will see that the following lemma is also quite easy to show which is that if E is a subset of the sub collection of the power set of x and such that phi and x both belong to this collection E. And rho is a map from E to the non-negative extended reals such that the empty set gives you the value 0 when you apply rho to it. So this is the empty set axioms. So rho is the any map need not be finitely additive but it is any map which satisfies the empty set axioms.

Then Mu star E define as the infimum of the values rho Ei i = 1 to infinity. Such that E is covered by this union i = 1 to infinity Ei and each Ei belongs to E is defines an outer measure on X. So this is exactly just a repeating the same arguments for the proofs that we gave after lebesgue outer measure that was define using the elementary measure and you can basically repeat each argument here to show that this is an outer measure.

So in fact when we define it using a pre measure of course the pre measure satisfies this empty set axiom. And the Boolean algebra satisfies this property that empty set and the whole set x belong to the Boolean algebra and so we can define an outer measure using this prescription. So by the theorem the caratheodory extension theorem implies that Mu star gives us a sigma algebra C Mu star X and a measure Mu which is the restriction of Mu star to this sigma algebra C Mu star X.

So once we have a premeasure we can define a outer measure from the outer measure we use the caratheodory extension theorem to get a measure. So now we have to show that this measure that we get extends Mu naught. So first we have to show that B naught the Boolean algebra is the sub collection of the sigma algebra C Mu star X. So let me denote it by B which will be the required sigma algebra.

And for any E in B naught Mu in B naught E is equal to Mu naught. This means that Mu when restricted to B naught gives you is given by the exactly the premeasures Mu naught. So we have to show this so remember that our sigma algebra was defined using the caratheodory measurability condition. So we have to show that if E is a element of B naught then Mu star then for any subset A of x Mu star of A is equal to Mu star A intersection E + Mu star A - E.

So this was the condition that was required to define the caratheodory measurability. So this will show that so this will show that E belongs to the sigma algebra B which was by definition this C Mu star of x. So since Mu star is an outer measure we have that Mu star A is less than or equal to Mu star A intersection E + Mu star A - E this is by sub additivity. This is finite sub additivity and of course finite sub additivity is a consequence of countable sub additivity and so we have this in equality.

So if suffices so if suffices to show that Mu star A is greater than or equal to mu star A intersection E + Mu star A - E. So let us use the fact that E belongs to this B naught so we start with an epsilon greater than 0 and now we are going to use a definition of the outer measure. So suppose that further that mu star A is finite. Of course if, mu star A is plus infinity this in equality of holds trivially so this is trivial if Mu star A is equal to plus infinity.

So we suppose that it is finite and so there exist this implies there exist a collection En in B naught such that A is covered by the union of this En’s and Mu star the sum n = 1 to infinity Mu star En is less than or equal to Mu star A + epsilon. So this we can do by the definition of so this should be mu naught here by definition of the outer measure that we have just defined.

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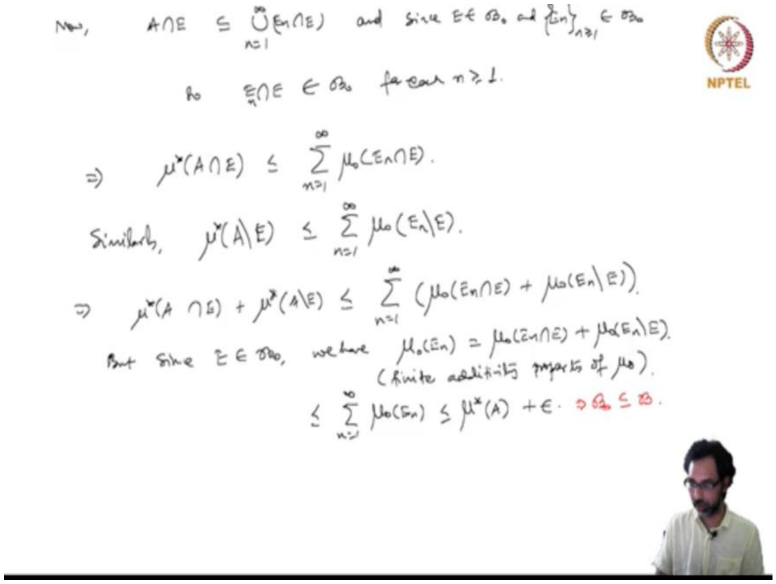

So now A intersection E is then covered by this union En intersection E n = 1 to infinity and since E belongs to B naught and En's belong to B naught so this correction is in B naught. So En intersection E belongs to B naught for each n greater than equal to 1. This means that the outer measure of A intersection E is bounded above by this infinite some of values Mu naught intersection E. Similarly we can show that Mu star of A - E is less than or equal to this sum n = 1 to infinity Mu naught of En - E.

So therefore if you add this to in equalities we get Mu star of A intersection E + Mu star of A- E is bounded above by the sum n = 1 to infinity Mu naught En intersection E + Mu naught En - E. Where, we have collected the nth term in the series in the 2 series in a single term. So we have a single sum with the nth terms given by the sum of the individual nth terms. But since E belongs to B naught we have that Mu naught En = Mu naught En intersection E + m=Mu naught En - E because this is a using the finite additivity property of the premeasure Mu naught.

So remember that Mu naught was a premeasure and to define a premeasure we add assume that we satisfy already a finite additivity property. So therefore this is less than or equal to the sum n = 1 to infinity Mu naught En and this is less than equal to Mu star A + epsilon. This is because this is how we chose our En so that the sum is bounded above Mu star A epsilon. So we are done and we have shown that B naught belongs to B naught is a sub collection of B.

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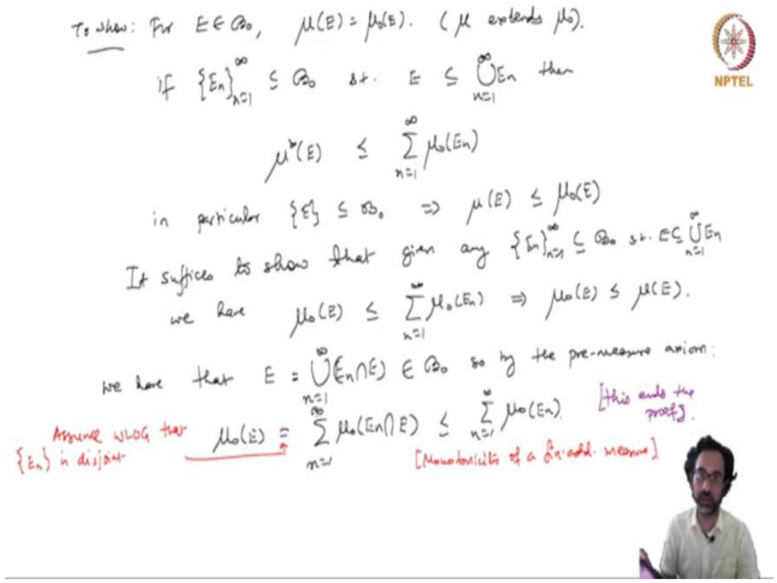

Now we have to prove that for so this is again to show that for E in B naught Mu E = Mu naught E. So, that our measure Mu that we got from caratheodory extension theorem when restricted to B naught is given by Mu naught. So Mu extends Mu naught so this means inverse that Mu extends Mu naught. So let see how this can be shown. So if En n = 1 to infinity is a collection of elements in B naught such that E is covered by this collection the union of this collection.

Then Mu star of E is a of course less than or equal to the sum of these Mu naught En n = 1 to infinity. So in particular you if you take the collection of this single set E this is the collection in B naught and if you want to take a countable collection you can take the rest of the set as empty. So then it implies that Mu star of E is less than or equal to Mu naught E because by the empty set axioms all are the terms are 0.

So we are automatically this bound which says that Mu star of E which is Mu of E because E belongs to B naught which we have shown that it belongs to the sigma algebra B. And we have this in equality. So it suffices to show that given any collection of this En's in B naught is such that E is covered by En's given any such collection we have mu naught E is less than or equal to this sum of Mu naught En's n = 1 to infinity. Because then you can take the infimum on the right side and this would imply that mu naught E is less than or equal to Mu E.

So why is this true? Of course we can have we have that E can be written as an union En intersection E n = 1 to infinity because En (()) (36:17) when you take the intersection becomes an equality rather than an inclusion. So this implies by and this belongs to B naught. So by pre measure axiom we have that Mu naught of equal to the sum n = 1 to infinity Mu naught of En intersection E.

And one can show that once you have a finitely additive measure it is automatically monotone I did not specify explicitly but this is by monotonicity property this is due to monotonicity of finitely additive measure. So one can show that once map satisfies the empty set axiom and the finite additivity axiom it also satisfies monotonicity axiom. So by monotonicity we have that each individual term is bounded above by Mu naught En and so we are done.

So this ends the proof so we have shown that just one more thing we have add here is that this equality here can be used. Because we can assume without loss of generality that is this correction En is disjoint. So in fact can write assume without loss of generality that En is a disjoint collection for example by reducing it to (())(38:25) so that the union gives you the same union but with disjoint spaces and then again you can use monotonicity.