Measure Theory
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Module No # 07
Lecture No # 34
Abstract measure and Caratheodory Measurability – Part 2

(Refer Slide Time: 00:22)

Closure under countable unions: Let $\{E_n\}_{n=1}^{\infty}$ be a collection in $\mathcal{C}_{\mu^*}(X)$. Suppose that it is a disjoint collection.

To show: $E = \bigcup_{n=1}^{\infty} E_n \in \mathcal{C}_{\mu^*}(X)$.

Let $A \subseteq X$. By countable sub-additivity

$$\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \setminus E).$$

It suffices to show: $\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \setminus E)$.

Now, $\bigcup_{n=1}^{N} E_n \in \mathcal{C}_{\mu^*}(X)$ for any $N \geq 1$. So, using the Caratheodory condition:

$$\mu^*(A) \geq \mu^*\left(A \cap \left(\bigcup_{n=1}^{N} E_n\right)\right) + \mu^*\left(A \setminus \left(\bigcup_{n=1}^{N} E_n\right)\right).$$

So now we can come to the closure under countable unions so let En n=1 to infinity be a collection in C Mu star x we suppose that this is a disjoint collection. Because even if it is not at disjoint collection we can rewrite the union as a disjoint collection by using the Lacuna formula and we can reduce to the case finite is the disjoint collection. So we will prove that to show that the union n = 1 to infinity En is Caratheodory measureable.

So let A be any subset and by countable sub addiivity we have that the Mu star of A is less than or equal to Mu star A intersection this E let me call this E. A intersection E + Mu star A – E so this always holds by countable sub-additivity of the outer measure Mu star. So it is suffices to show it suffices to show that Mu star of A is greater than or equal to Mu star A intersection E + Mu star A – E.

Now to show this we know that any finite union of En’s belongs to C Mu star x for any N greater than or equal to 1. So this is by the finite closer and the finite unions so using the Caratheodory property condition we get that Mu star of A is greater than or equal to Mu star of A intersection this finite union + Mu star of A minus this finite union. Now we can compare the second terms in this inequality and in this inequality.

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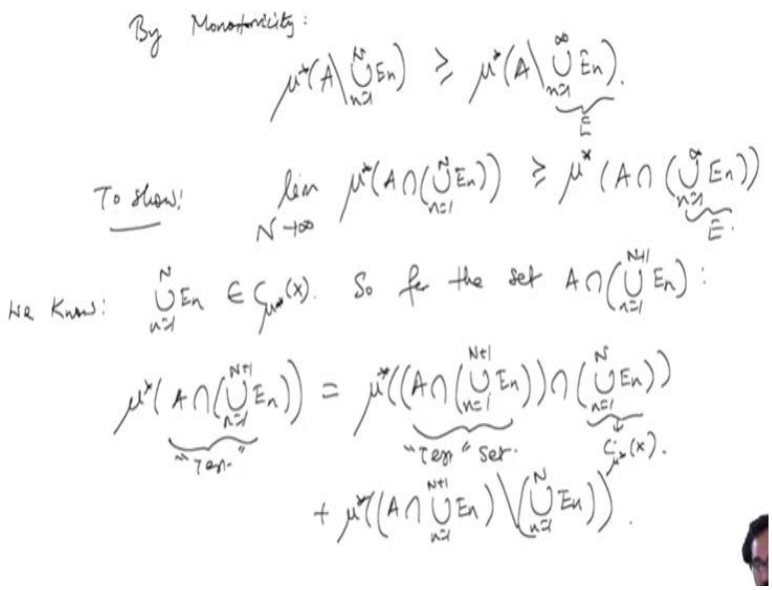

So by monotonicity property we get that Mu star of A- this union finite union is greater than or equal to Mu star of A- the infinite union. So this is again E so the second term here is greater than or equal to second term here so only need to show that the first term is greater than or equal to the first term here. So to show that as we take the limit as N goes to infinity f Mu star intersection of the union this is N going to infinity of this En's this is greater than or equal to Mu star of A intersection in a whole union from 1 to infinity this is E.

So to show this we use again that this union n =1 to N En we know that this belongs to C Mu star x. So for the set A intersection n = 1 to N+1 En we have Mu star A intersection union n = 1 to N + 1 En this is equal to Mu star a, intersection union N + n from 1 to N+1 En intersection with the union n = 1 to N En. So this is our test set and this is our set which belongs to C Mu star of x. This is a test set which means that this place the role of the Caratheodory condition for arbitrary sets. So this is the first term and the second term is A intersection union n = 1 to N + 1 En –n = 1 to N En.

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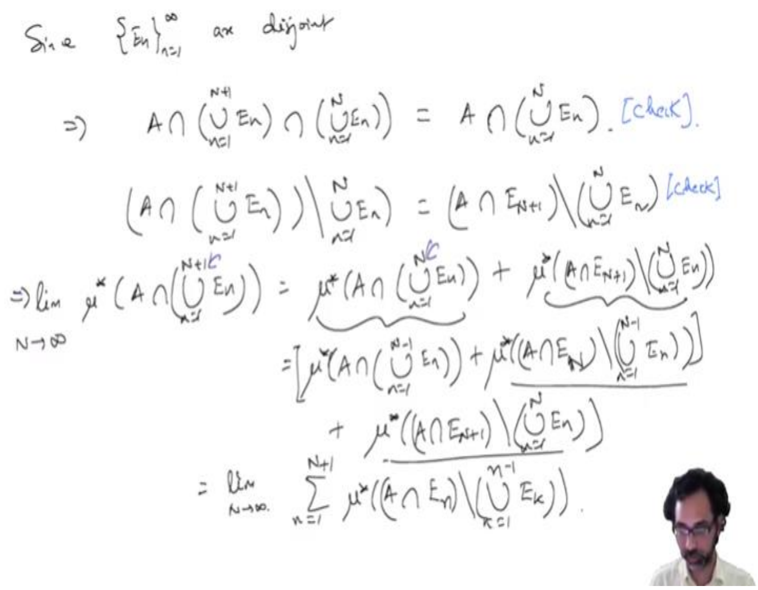

So we will we can analyze what these 2 sets are so because since this En's are disjoint from each other this means that A intersection this union n = 1 to N + 1 En intersection with n = 1 to N En. This is nothing but A intersection union of n =1 to N En so check this result holds check this and the second term is A intersection union n = 1 to N + 1 En – union n = 1 to N En. A intersection En + 1 – this union n = 1 to N of the En's so again check both this conditions check that they are valid.

So then we get on the left side Mu star of A intersection union n =1 to N + 1 En = Mu star A intersection n=1 to N En + Mu star A intersection En + 1 – n = 1 to N En. So now this is a recursive formula because here we have n + 1 and here we have N. And so we can rewrite again this term with n – 1 so this is equal to Mu star A intersection N -1 En + Mu star A intersection En -1 union n = 1 to N-1 En.

So this is simply the first terms here and plus we have the second term here A intersection En + 1 – union n= 1 to N En. So we see that we will get terms of this type repeatedly and so this is in fact the sum n = 1 to N + 1 Mu star of A intersection En – union k = 1 to, n -1 Ek. And now we can take the limit as N tends to infinity on both sides.

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$$\lim_{n\to\infty} \mu^*\left(A \cap \bigcup_{k=1}^{n} E_k\right) = \sum_{n=1}^{\infty} \mu^*\left((A \cap E_n) \setminus \left(\bigcup_{k=1}^{n-1} E_k\right)\right)$$

On the other hand

$$\mu^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) \leq \mu^*\left(\bigcup_{n=1}^{\infty} \left((A \cap E_n) \setminus \bigcup_{k=1}^{n-1} E_k\right)\right).$$

$$\leq \lim_{n\to\infty} \mu^*\left(A \cap \left(\bigcup_{k=1}^{n} E_k\right)\right)$$

$\Rightarrow$ $\mathcal{C}_{\mu^*}(X)$ is closed under countable unions.

To show: μ^* restricted to $\mathcal{C}_{\mu^*}(X)$ is a measure.

So on the left hand side we get limit n tends to infinity Mu star of A intersection k from 1 to n Ek and on the right hand side we get k = 1 to infinity Mu star rather n = 1 to infinity A intersection En – union k = 1 to n – 1 E. On the other hand we also have that the Mu star A intersection the entire union from k =1 to infinity Ek this is less than or equal to this sum because we have Mu star of this is the union of n = 1 to infinity of the sets.

So A intersection En – union k = 1 to, n -1 Ek so this is less than or equal to this sum which is nothing but the limit on the left hand side Mu star of A intersection k = 1 to n Ek. And this is what we wanted to prove. So we have shown that this implies that C Mu star of x is closed under countable unions. So now we have to show that Mu star restricted so again to show restricted to C Mu star x is in fact a measure.

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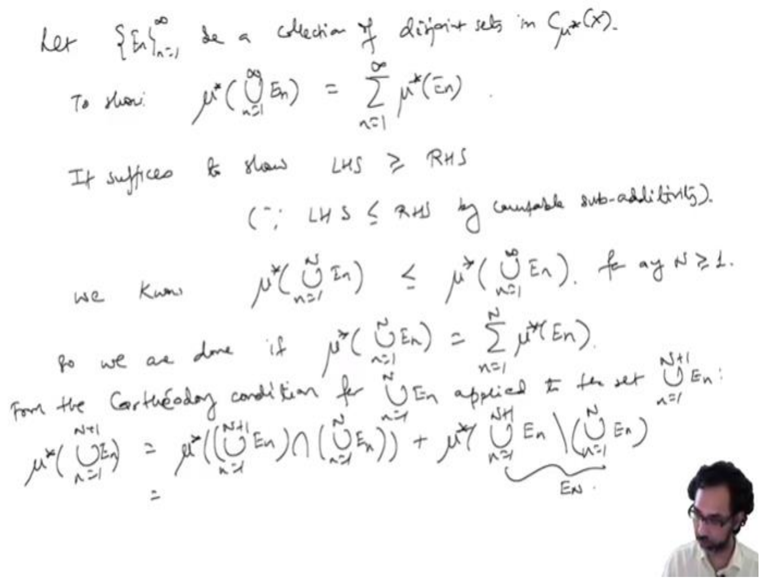

So then let Ek or En be a collection of disjoint sets in C Mu star x and we have to show that Mu star of the union n =1 to infinity En is equal to the sum Mu star En n = 1 to infinity. So again it is suffices to show only that the LHS is greater than or equal to the RHS because LHS is less than or equal to RHS countable sub additivity. So to show this we know that the Mu star of any finite union of En's is less than or equal to the Mu star of the infinite unions.

So we have done so this is for any N greater than equal to 1 so we have done if Mu star of n=1 to N En is equal to the sum n =1 to N Mu star of En. So this is the finite additivity rule but since we do not know that this outer measure satisfies this finite additivity rule we have to prove it. So from the Caratheodory measurability, Caratheodory condition for union of n =1 to N En applied to the test set the set union n =1 to N + 1 En.

We have that Mu star of n = 1 to N + 1 En = Mu star n =1 to N + 1 En intersection with the union n = 1 to N En + Mu star union n = 1 to N + 1 En – n = 1 to N En. So this is nothing but EN and this is nothing but so let me write it in another page.

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$$= \mu^*\left(\bigcup_{n=1}^{N} E_n\right) + \mu^*(E_{N+1}).$$

By induction $\Rightarrow$ $\mu^*\left(\bigcup_{n=1}^{N+1} E_n\right) = \sum_{n=1}^{N+1} \mu^*(E_n)$

this concludes the proof.

So this is nothing but n = 1 to N En and the second term is nothing but EN+1. So by induction this implies that Mu star of union n = 1 to N+1 En is the sum n = 1 to N + 1 Mu star of En and we have done. This concludes the proof.