Measure Theory
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Module No # 07
Lecture No # 33
Abstract measure and Caratheodory Measurability – Part I

So in the last lecture we have seen the definition of Boolean algebra and sigma algebra ad we have seen examples of both kinds of algebras. And in this is lecture we will look at abstract outer measures and the notion of measurability on abstract spaces. So this I have turned as Caratheodory measurability and generalize the notion of the Lebesgue measurability that we know for Rd.

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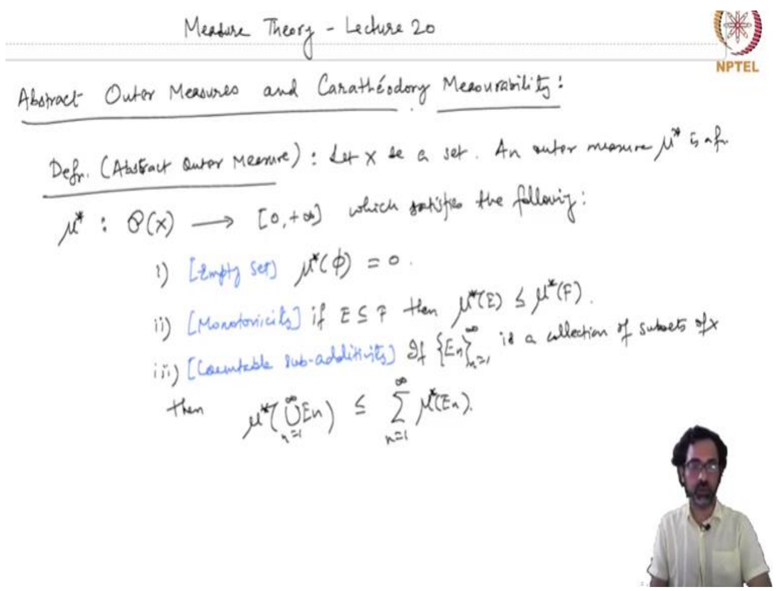

So let me define then notion of abstract outer measure and abstract measure for an arbitrary subset x. So let x be a set then an outer measure which I will denote by Mu star it is a function Mu star is a function from the power set of x to the extended positive real lines 0 to + infinity which satisfy the following axioms. So first is that the empty set axiom so these are the same outer measure axioms that we have stated for the Lebesgue outer measure.

So the first one is that the outer measure of the empty set should be 0 the second is the monotonicity property. Which states that if E is the subset of F then Mu star E is less than or equal to Mu star F and third one is countable sub-additivity property. Which states that if En n=1 to infinity is a collection of subsets of x. Then the outer measure of the union of the En's is bounded above by the sum n=1 to infinity Mu star of En.

So these 3 properties axioms will constitute what is called as abstract outer measure on the set X. So now let me define the notion of an abstract measureable space.

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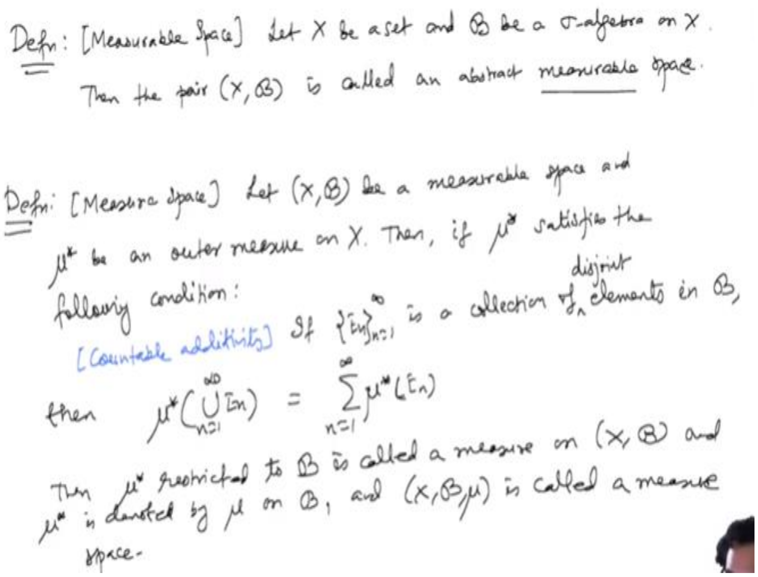

So this let x be a set and B be a sigma algebra on x so we have seen a definition of sigma algebra in the last lecture. So we are telling an x we are taking a set x and sigma algebra on x then the pair x, B is called a measureable space is called an abstract measureable space. So here we are using the term measureable and once we have defined the notion of an measure on a measureable space.

Then we will call it a measure space so let me define now what is a measured space? So let x be a measureable space and Mu star be an outer measure on x. Then if Mu star satisfy the following condition which is that so Mu star is already an outer measure. And it should now satisfy the condition of countable additivity property which is that if En's n = 1 to infinity is the collection of elements in this sigma algebra B then collection of disjoint elements in B.

Then the outer measure for the union of all these En's should be the sum exactly the sum of the outer measures of En's. So if an outer measure on a measureable space satisfies in addition is countable additivity property. Then Mu star restricted to B is called a measure on x, B and the triple x. So now first let me write that Mu star is denote by simply Mu on B and the triple x, B Mu is called a measure space.

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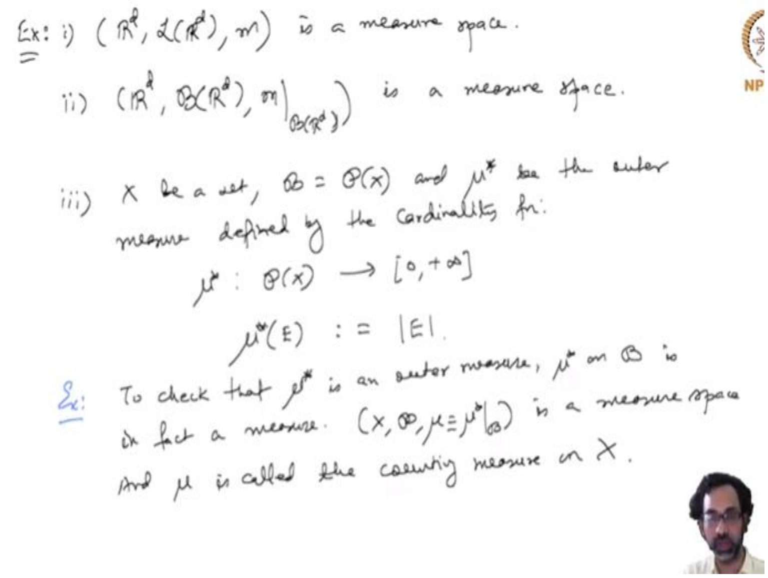

So of course our prototypical example for a measure space is Rd with the Lebesgue sigma algebra and so this is our first example and the Lebesgue measure this is a measure space. Similarly if you take Rd with the Boral sigma algebra and m restricted to the Boral sigma algebra then also it will satisfied the countable additivity property and so this is also a measured space. Now for an abstract example let us take a x be a set and B be the power set of x and Mu star be the outer measure defined by the cardinality function.

So Mu star is a map from the power set to the positive non-negative extended real line which takes any subset E of x and it is by definition that cardinality of E. So of course if E is a countably infinite or uncountable then this will take the value plus infinity here. But if it is finite then it will give you a finite value now B is sigma algebra and Mu star thus satisfy the countable additivity.

So I leave it to you as an exercise to check that first that Mu star is an outer measure by using the definition of cardinality of sets that we have seen before. And secondly that Mu star on this whole sigma algebra the discrete algebra B is in fact a measure. So in this way x, B, Mu here is again the restriction of Mu star on B is the measure space. And here Mu is called the counting measure on x.

So this is an example of an abstract measure space which is just by looking at the cardinality function on any subset of x. So now that we have a notion of an outer measure and a measure we would like to know when we have an outer measure whether we can upgrade it to a measure.

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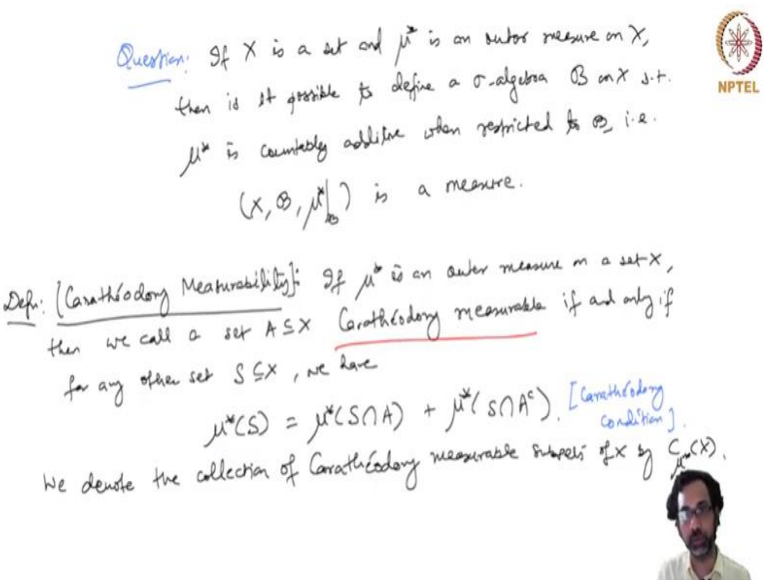

So question is that if x is a set and Mu star is an outer measure on x then is it possible to define a sigma algebra B on x such that Mu star is countably additive when restricted to B which in other words means that x, B and the restricted of Mu star to B is the measured space. So once you have outer measure on a set x we would like to know when we can upgrade it to a measure and this is answered by the notion of Caratheodory measurability condition.

So this is the definition and we recall that we had the Lebesgue outer measure in Rd and then we define the sigma algebra of Lebesgue measureable sets by using the notion of almost open or outer approximation by an open set or an inner approximation by a closed set. But here we do not have the notion of an open or a closed set because x may not even be a topological it is an abstract set.

So we need to find an equivalent condition which would work for Rd it should give us for the Lebesgue outer measure again the Lebesgue measurable sigma algebra of Lebesgue measureable sets. But this definition should also work for any set x so this is what is known as Caratheodory measurability and this is the if Mu star is an outer measure on x on a set x then we call a set A of x caratheodory measureable.

If and only if for any other set S of x we have that Mu star of the set S is equal to Mu star of S intersection A + Mu star of S intersection A compliment. So this is called the caratheodory condition and if it this condition needs to be satisfied for any arbitrary subset S of x then A is called caratheodory measureable. And we denote the collection of caratheodory measureable subsets of x / C Mu star of x.

So of course we have to now show that our notion of measurability satisfies the expected properties which are that it the collection of measurable subset caratheodory measurable subsets should give you sigma algebra. And Mu star restricted to that sigma algebra should be countably additive.

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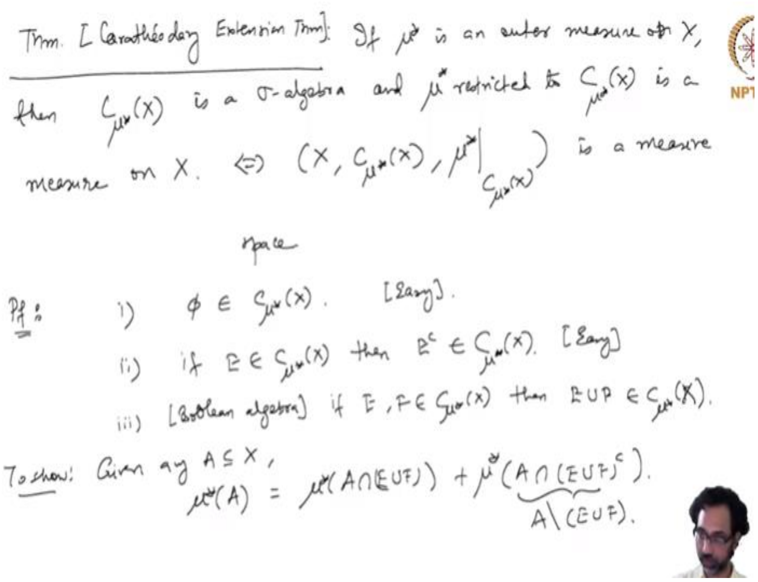

So this is the famous caratheodory extension theorem and this says that if Mu star is an outer measure on x. Then C Mu star x the collection of caratheodory measureable subsets of x is the sigma algebra and Mu star restricted to this sigma algebra is a measure on x. Meaning that it is satisfies countable additivity property as well so this says that this answers our question is that once you have an outer measure by the caratheodory measurability condition you automatically get a sigma algebra and a measure on that sigma algebra.

So equivalently x with the sigma algebra C Mu star X and the restriction of Mu star to C Mu star x this is a measure space. So once you have an outer measure you automatically get a measure. So let us see the proof which is a bit long but we will go step by step so the first one is that Phi belongs to C Mu star x. So we need it that the empty set should belong to the sigma algebra and this is obvious this is easy so I will leave it as an exercise to check for you.

The second one is that if E belongs to C Mu star x then E complement also belongs to C Mu star x this is almost immediate from the definition of the caratheodory measurability. Because it is symmetric with respect to taking compliments so this is also easy and I will leave it for you to check. So the third one is to check whether it is a Boolean algebra so if E and F are in C Mu star x then E union F should be in C Mu star x.

So we have to show this so to show given any A any subset of x we have to show that an outer measure of A is equal to the outer measure of A intersection E union F plus the outer measure of A intersection E union F complementary. So this is nothing but A – U union F so we have to show this for any subset A of x so this will prove that if E and F are Caratheodory measureable then their union is also Caratheodory measurable.

And by induction any finite union of Caratheodory measureable sets will be Caratheodory measureable and therefore it will be a Boolean algebra. So let us try to show that this whole.

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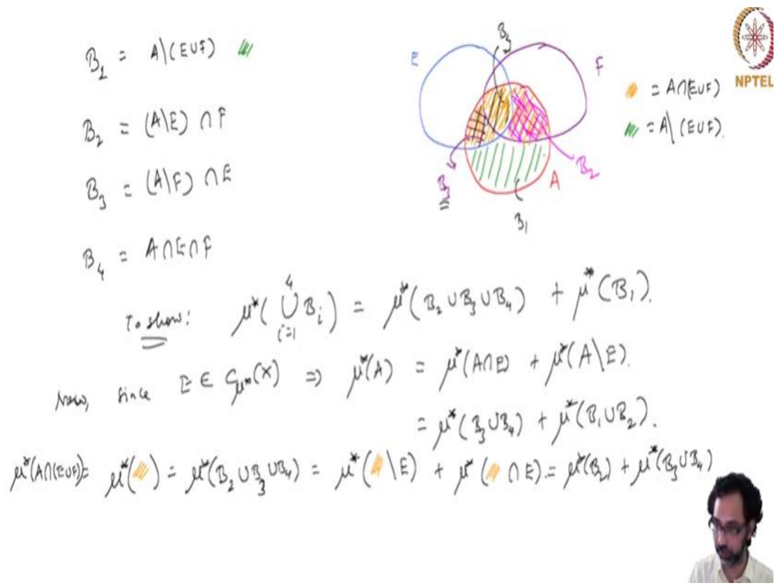

So here we have a Venn diagram of this situation we have E and F and then A which is an arbitrary subset. And we have the outer measures of 2 parts one is A intersection E union F in the orange shade and A – E union F in the green shade and I have divide this relevant parts into 4 subparts B1, B2, B3, B4. B1 is nothing but this green part so this is the green part B2, B3 and B4 are the 3 parts of the orange shaded region.

So for example B2 is A – E intersection F which means that this region here this Magenta region this is B2. Similarly B3 here this is B3 and the intersection of A, E and F the middle part this is B4. So we see that we have to show that Mu star of A which is the union of this disjoint sets Bi i = 1 to 4 = Mu star of the orange shaded region which is precisely B2 union B3 union B4 + Mu star of B1.

So I have so we stated our goal in terms of subsets this B1, B2, B3 and B4 now we use the fact that E belongs to Caratheodory measurable E is a Caratheodory measureable set. So this implies that a Mu star of A so the Caratheodory condition applied to A = Mu star of A intersection E + Mu star of A – E. But Mu star of A intersection E is nothing but the union of B3 and B4 here. So this is B3 union B4 + Mu star of A – E which is this is B1 union B2.

But if we apply on the other hand if we apply this is the same condition this is the Caratheodory measurability of E for the set which is given by the orange shaded region which is again the union of B2, B3 and B4. This is by the way this is same as Mu star of A intersection E union F and so this is nothing but the Mu star of the shaded region –E or in shaded – E+ Mu star of orange shaded region intersection with E.

So this is nothing but the first one is simply the Mu star of B2 which is this part here in magenta this is B2 and the second part is B3 union B4 which is these 2 parts here. So we have that Mu star of the orange shaded region is Mu star B2 + Mu star B3 + B4.

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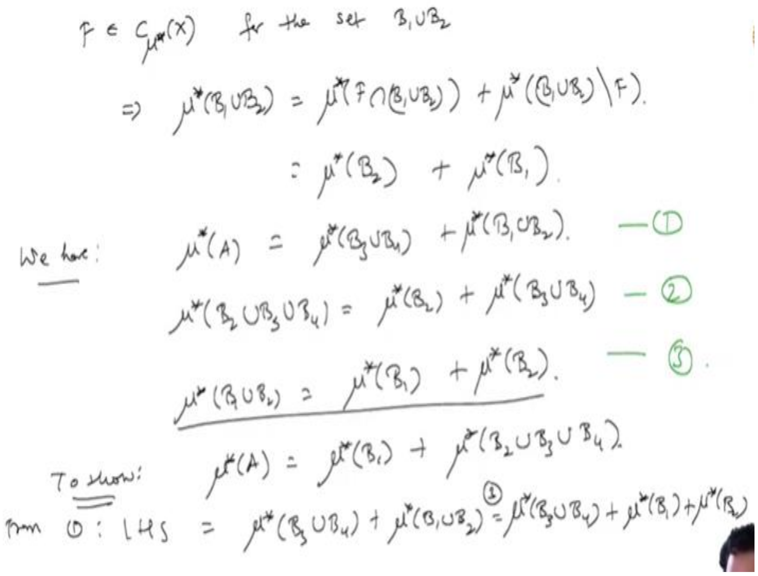

Now I am going to use the fact that F also belongs to the Caratheodory measureable sets. So I am going to apply it for the set B1 union B2. So this implies that Mu star of B1 union B2 = Mu star of F intersection B1 union B2 + Mu star of B1 union B2 – F. So let us see what this sets are so the first one is B1 union B2 intersection F. So this is B1 here in the green shaded region B2 is the magenta shaded region so B1 union B2 intersection with F is simply B2.

So we get Mu star of B2 and the second one is B1 union B2 –F but B2 is a subset of F so you will simply get B1. So you will get Mu star B2 of Mu star of B1 so now we have 3 equalities we have first is that Mu star of A = Mu star of B2 union B3 union B4 + Mu star of B1 so this is the first one this is what we proved. Let us collect the equalities that we have got. The first one is Mu star A = Mu star B3 union B4 + Mu star B1 union B2 so this is this one the second one we got is Mu star B2 union B3 union B4 = Mu star B2 + Mu star B3 union B4.

And the third one that we got is Mu star of B1 union B2 = Mu star B1 + Mu star B2. And we had to show remember that Mu star of A = Mu star of B1 + Mu star of B2 union B3 union B4. So on the left hand side they get so let me number these equalities 1, 2 and 3. So for the left side we get from 1 we get that the left side is equal to Mu star B3 union B4 + Mu star B1 union B2 which is equal to Mu star B3 union B4 + Mu star B1+ Mu star B2 this is from the third equality here.

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For RHS: $\mu^*(B_1) + \mu^*(B_2 \cup B_3 \cup B_4)$

$$= \mu^*(B_1) + \mu^*(B_2) + \mu^*(B_3 \cup B_4) \quad \text{from } ②.$$

So $\mu^*(A) = \mu^*(A \setminus (E \cup F)) + \mu^*(A \cap (E \cup F))$.

(this holds for the case $\mu^*(A) < \infty$ or $\mu^*(A) = +\infty$).

$\Rightarrow$ $C_{\mu^*}(X)$ is a Boolean algebra.

On the other hand for the RHS we get RHS we get Mu star B1 + Mu star B2 union B3 union B4 this is equal to Mu star B1 + Mu star B2 + Mu star B3 union B4 from the second equality. So we see that the LHS and the RHS are the same and therefore we get the result. So Mu star A = Mu star A – AE union F +Mu star A intersection E union F. Note that we did not use the fact that any of these outer measures are finite or infinite and we did not use any cancellation so this holds for the case Mu star is A is infinite or Mu star A.

So this proves that finite unions are inside the Caratheodory measureable collection which means that C Mu star X is a Boolean algebra.