Measure Theory
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Module No # 07
Lecture No # 32
Abstract measure spaces: Boolean and Sigma-algebras

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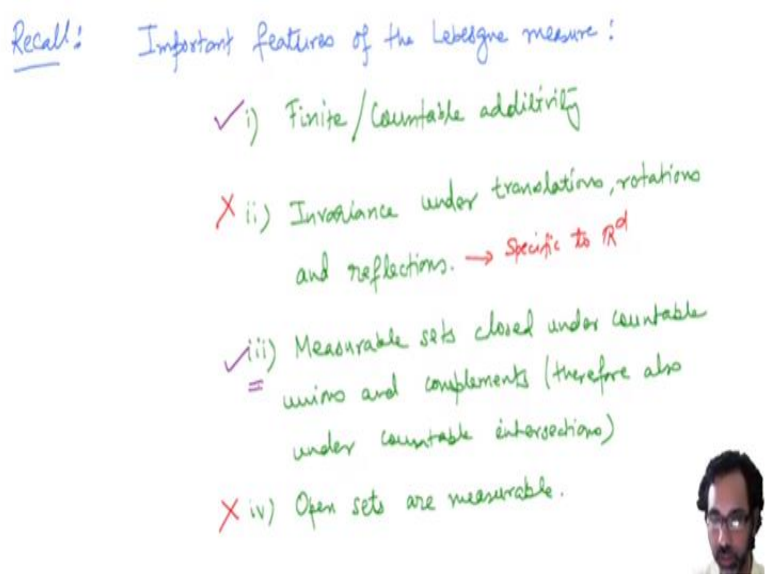

Let us review some of the important features of the Lebesgue measure the first one is countable and finite-additivity property which we have seen holds for Lebesgue measureable sets buts fails in general for sets which are not Lebesgue Measurable. The second is invariance under translations rotations and reflections which implement our geometric intuition about how a measure should behave for subsets of the Euclidean space Rd.

The third is that it is collections of measurable sets are closed under countable unions and complements. And so they are also closed under countable intersection as well. And the fourth is that all open sets are measureable so it gives us a connection with the topology of the underline space in this case which is Rd it gives us that all open subsets of Rd are Lebesgue measurable. So now the goal is to pass to sets other than Rd and see whether the concept of a measure can be implemented for arbitrary spaces.

And for this purposes notice that any other set x may not even have a notion of rotation on that set or a reflection or a translation. So these were specific to Rd so this is specific to Rd that we have used and so we are going to drop this requirement that it should be in varied under translations, rotations and reflection. So it might happen that your space is linear and it does make sense to talk about translations, rotations and reflections.

But we are aiming for measure theory for abstract sets so we have to drop this condition in order to define a concept of measure for abstract sets. On the other hand the fourth one is a connection with the topology of the underline set. But again we are not imposing any existence of topology for our sets so it may not be a topological space it can be an arbitrary set as well. So the connection with the topology also as to be dropped so what we are going to keep is the first one which is finite or countable additivity.

And that it should be closed under countable unions and a complement and so it will also be closed under countable intersections. So the first and the third property make sense for an arbitrary set which need not be an arbitrary space like Rd or which need not be a topological space. So we are going to keep this as part of actions of our axioms for measures on arbitrary sets. So of course measure on arbitrary sets may not be defined as the whole on any subsets of that arbitrary sets.

So as we have seen in Rd not all sub sets are measurable so we have to impose some condition on what our measureable sets should look like. And the prototypical example the collection of Lebesgue measureable sets will give us the way forward on how to think about thing. So this is property 3 which says that it should be closed under countable (()) (04:27) and complements. And I have mentioned in before that these are part of what are called as sigma algebra. These are axioms for sigma algebras.

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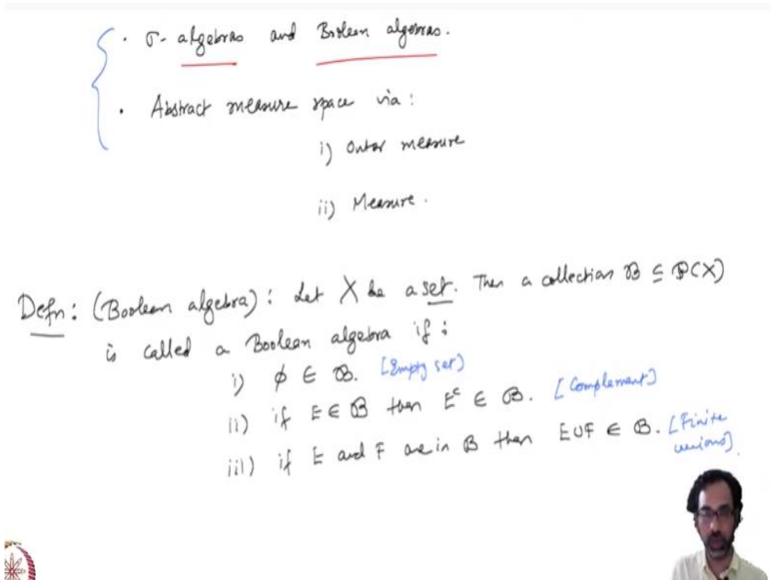

So today in this lecture we will see that the concept of sigma algebras and Boolean algebras. And we will also define the notion of an abstract measure space via the concept of an outer measure and abstract outer measure and then abstract measure. So this measure will be only restricted to sigma algebra's that they are going to associate with this abstract set and we will call the elements of those sigma algebra's measureable.

Like we did for Lebesgue measure so in this lecture we will try to cover these 2 concepts so let us starts with the notion of a Boolean algebra and then we will pass to the sigma algebra. So let us define what is a Boolean algebra? And abstract Boolean algebra that is so let x be a set so like I said we are trying to develop a theory of measures on abstract sets which may not be Rd which may not be even a topological set space.

So we are only measuring that it is a set so x is the set then a collection let me denote it by the B the spelling is B which is a subset of the power set of x is called a Boolean algebra if so first is that empty set should belong to the. Secondly if a set belong to the then it is compliment should also belong to the entered that if we have 2 sets E and F are in B then the union E union F should also be on to B. So the empty set axiom then we have a closed under compliment and closed under finite unions. So these 3 axioms may come the structure of the Boolean algebra.

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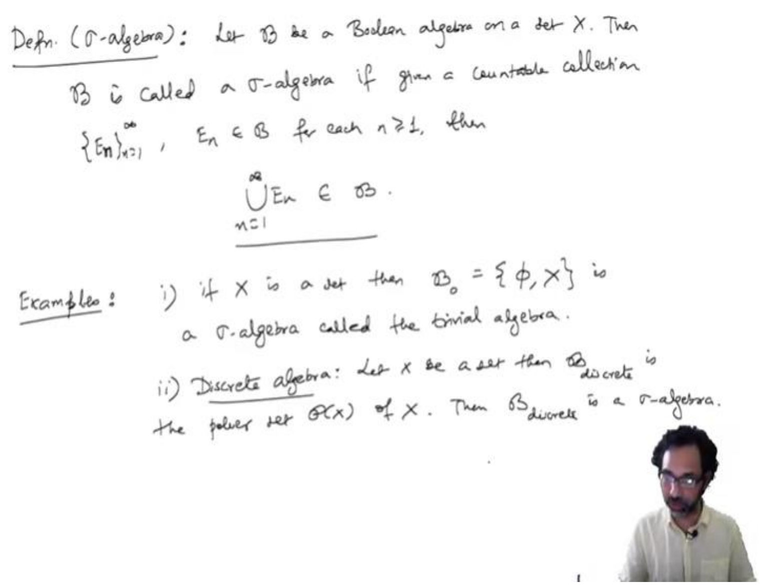

The next definition is for a sigma algebra so let B be a Boolean algebra on a set x then B is called a; sigma algebra. If given a countable collection En n = 1 to infinity each En belongs to B for each n greater than equal to 1. Then the union of all these En's n = 1 to infinity also belongs to B. So it is a sigma algebra is a Boolean algebra which is in turn in addition closed under countable unions.

So this is extra condition that we will for a Boolean algebra to be a sigma algebra now let us see some examples of Boolean and sigma algebra. So the first one is if x is a set then a B naught given by a collection empty set and the whole set x is a sigma algebra called the trivial algebra. It is quiet easy to show that it is a Boolean algebra because the compliment of the empty set is and the compliment of x is the empty set and of course finite unions belong to the same set same collection.

And of course countable unions will also belong to the same collection because countable union of let us say empty sets is empty and countable union of x is x. If you have x union Phi then it is still x and so on so it is quiet easy to show that this is in fact sigma algebra. The second one is called the discrete algebra and the discrete algebra is given by on the set x. So again let x be a set then B discrete is the power set Px of x so you consider the entire collection of subsets of x then it gives you a; sigma algebra.

Then B discrete is sigma algebra so of course this is obvious because you only have complements and countable unions and each such is a subset of x and so it is trivially a sigma algebra which is called the discrete algebra.

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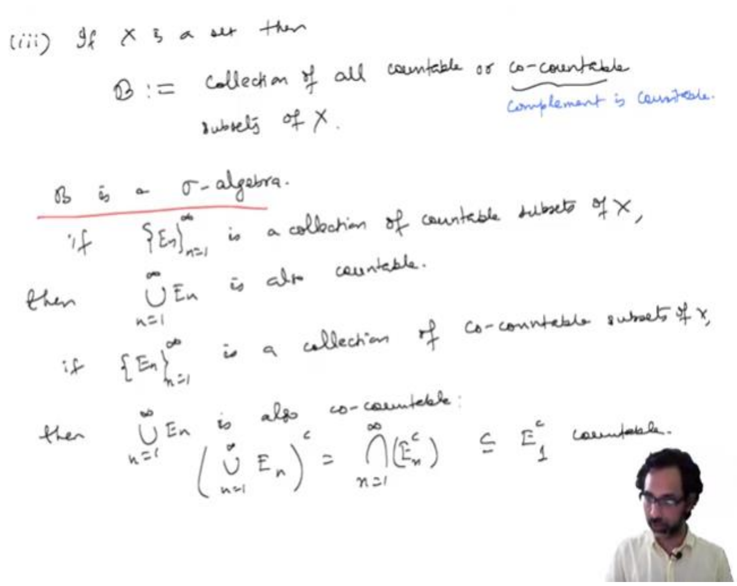

The third interesting example is if x is a set then you can define B to be the collection of all countable or co-countable subsets of x. So countable subsets which are in bijection with natural numbers and co-countable this means that the complement is countable. So this is in fact B is a sigma algebra this is because if let us say E and n = 1 to infinity is a collection of countable subsets of x only countable and no co-countable subsets of x.

Then the union n = 1 to infinity is also countable but countable union of countable sets countable so therefore the union of E and C is also countable. Similarly if En n=1 to infinity is a collection of co-countable sets subsets of x then the union n = 1 to infinity En is also co-countable because the complement of the unions of En this complement this is the intersection by De Morgan Law of the complements.

And so it is in particular the sub set of E1 complement and this is a countable set so subset of a countable set must be a countable. So this is countable so the union is co-countable and if you have mix it you can separate out the countable and co-countable sets. So you will still get a union of countable and co-countable subsets of x and so this collection B is sigma algebra. So this is a quite easy of course if you take complements it is stabilize because we are taking both countable and co-countable.

So if you take complements it is still closed under taking complements so it is this sigma algebra. Now let us look at some examples which we have already seen.

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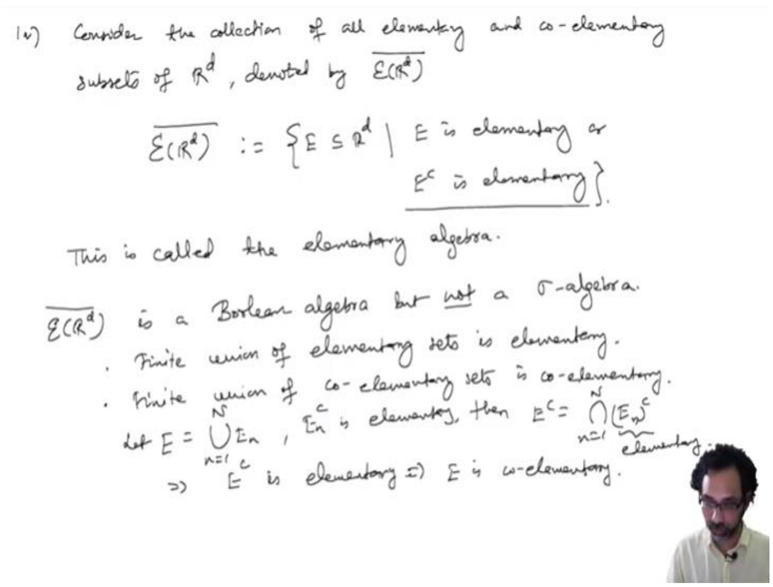

So consider so this is the fourth example consider the collection of all elementary and co-elementary subsets of Rd. So this means that so I will denote it by ERd bar. So ERd whole bar is the subsets of Rd such that E is elementary or E complement is elementary. So this is called the elementary algebra so we will see that this is closed under finite unions but not closed under countable unions.

So of course it is closed undertaking complements but it is not closed under taking countable unions. So in fact is collection is a Boolean algebra but not a, sigma algebra so of course at finite unions of elementary sets is an elementary. A finite union if you take so let me go step by step first is that a finite union of elementary sets is elementary. Second is that finite union of co-elementary sets so these are precisely subsets whose countable complement is elementary co-elementary sets is co-elementary.

This is because if you take a En's n=1 to N so let me write E for this finite union then each is En complement is elementary. Then E complement is the intersection of the complements of En's and these are called elementary and finite intersections, of elementary sets is elementary. This implies that E is elementary so it is closed under complements fine and finite unions. If you take finite unions of elementary sets you can get an elementary set if you get co-elementary sets if you get a co-elementary.

So you get E complement is an elementary means E is co-elementary and of course the empty set is elementary. So this gives your Boolean algebra but I also claimed that it is not sigma algebra so this is a quite easy.

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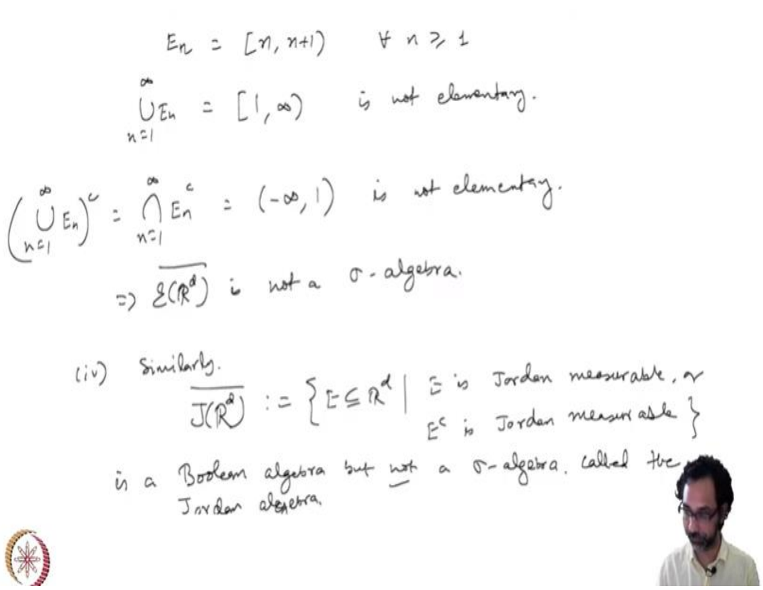

So you can take En to be n, n+1 for all n greater than equal to 1 let us say. So the union is of course not elementary union En, n = 1 to infinity is just the set 1 to infinity and this is not elementary it is unbounded. So it is not elementary and it is not also co-elementary because the intersection of the complements so this is the complement of the intersection or complement of the union. And this is simply the set minus infinity to 1 and this is also not elementary this is also unbounded and it is not elementary.

So the union is neither elementary nor co-elementary which means that B is not sigma algebra sorry I denoted it as ERd bar. So it is a closer of the elementary sets by taking co-elementary so that is why there is an over line here and this is not a sigma algebra. Similarly the collection JRd bar which is the collections of subsets of Rd such that E is Jordan measureable either E is Jordan measurable or E complement is Jordan measureable.

So this is again a Boolean algebra one can check that this is a Boolean algebra but not sigma algebra this is called Jordan algebra. So the Jordan algebra is a Boolean algebra but not a, sigma algebra.

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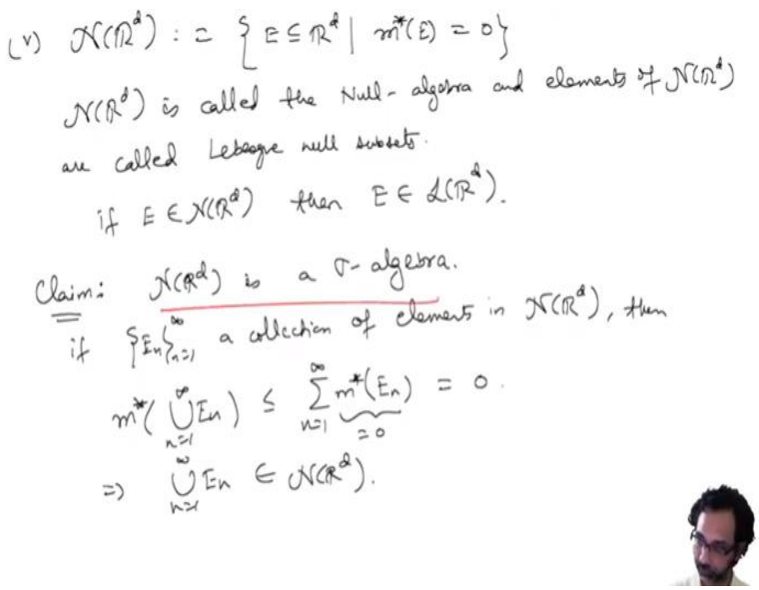

The next example is so called null algebra which is by definition the collection of subsets of Rd whose outer Lebesgue measure is 0. So this is the so called NRd is called the null algebra and elements of Nd in Rd are called Lebesgue null subsets. So of course if outer measure is 0 we have seen is Lebesgue measureable so if E belongs to NRd then E belongs; the collection of Lebesgue measurable sets of Rd.

So I claim that this NRd is a sigma algebra so this is not too hard so if you take En n=1 to infinity or collection of elements in NRd then the outer measure of the union is bounded above by the sum by countable sub-addivitity. And each of them are 0 because each En is a null set so each of them is 0 so this is also 0 which implies that the union En belongs to NRd. Of course the empty set is also belongs to NRd and if you take finite unions also it also belongs to NRd. So this implies that NRd is a sigma algebra called null algebra.

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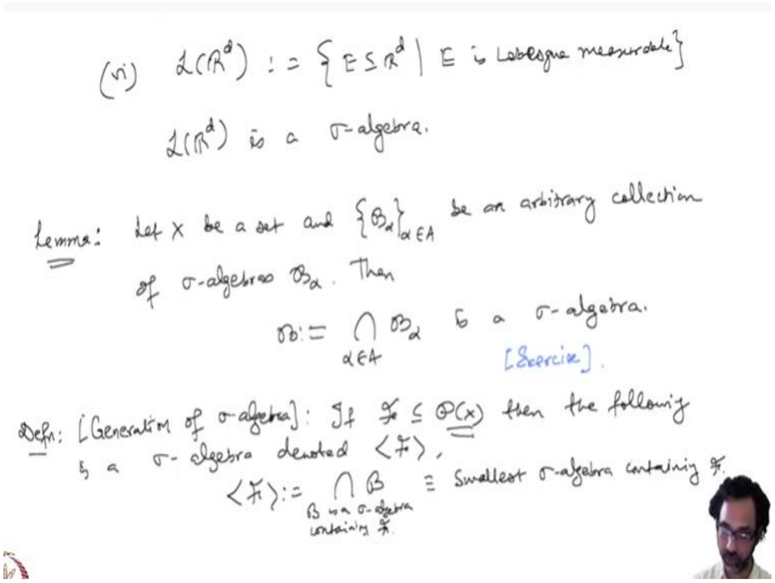

Another example is LRd itself this was the collection of Lebesgue measurable subsets of Rd and we have seen already that LRd is the sigma algebra. It is closed under taking complements and it is a closed under taking countable unions forget to mention that if you take NRd as simply those subsets which are Lebesgue outer measure 0 then it may not be closed under complements. So one has to add that it is a e-complement either E as Lebesgue measure 0 or e-complement as Lebesgue measure 0.

So one has to check a further property here that if you take the elements of NRd for which m star E = 0 are called Lebesgue null and if m star e-complement is 0 they are called Lebesgue co-null. So if you take En all co-null then I claim that the unions of all this co-nulls is also co-nulls because if you take the complement then you get the intersection of all this co-null for the complement of the co-null sets, and this is less than or equal to m star E1 complement and this is 0.

So again you get co-nulls so you see that if you take a, null sets you get nulls so here one has to take null sets and if you take co-nulls then null set get the co-null sets. So in case of Lebesgue measurable sets it is already closed under complement and we have seen that it is also closed under countable unions so it is sigma algebra. Now one property so I will state it as Lemma is that if B alpha so let x be a set and B alpha EA be an arbitrary collection of sigma algebra’s B alpha.

So each B alpha is sigma algebra then the intersection of all, this B alpha's so also sigma algebra. So this says that arbitrary collection of the intersection of arbitrary collection of sigma algebras is sigma algebras. So, I leave the proof as an exercise quite easy so with this we can make the following differential which is called the generation of a sigma algebra. This is that if f is the collection of subsets of x then so then the following is a sigma algebra denoted by this F with this pointed brackets.

And this F is nothing but the intersection of all sigma algebra's so let me write the intersection of B is a sigma algebra containing F containing all elements of F. So this is a smallest sigma containing F so we call this the sigma algebra generated by F note that this intersection is always non empty because the power set itself a sigma algebra containing F so this is always non empty.

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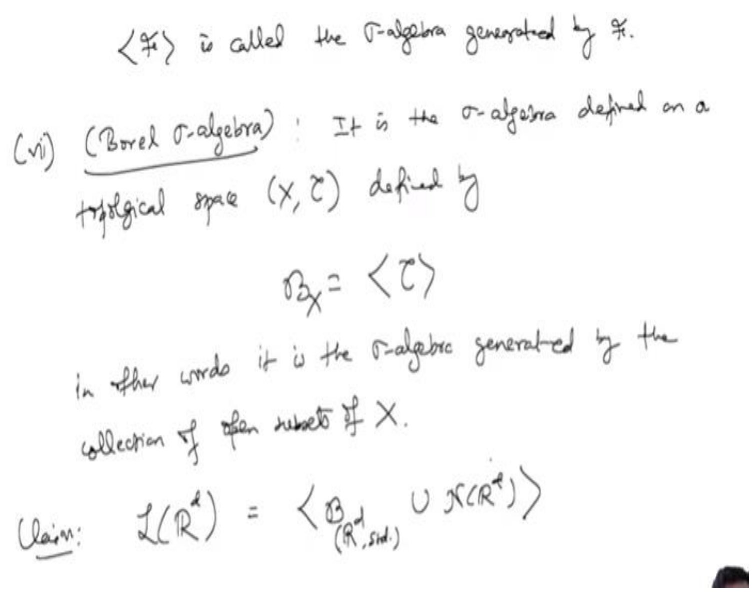

And F is called the sigma algebra generated by the collection F so we can give another example so I was 6 already so 7 example. This is a very important class of sigma algebra called the Borel sigma algebra which is defined as it is the sigma algebra. Defined on a topological space x with the topology tou defined by B is the sigma algebra generated by the topology tou or in other words it is the sigma algebra generated by the open sets of x the collection of open subsets of x.

So I will write Bx when I have to specify that it is for R for a specific set x now this is a sigma algebra by definition it is generated sigma algebra. So now I will claim that the collection of Lebesgue measureable sets is generated by the Borel sigma algebra on Rd meaning this is for the standard topology on Rd. So for the standard topology and union with the null algebra, so this says that the Lebesgue measurable the sigma algebra of Lebesgue measureable sets is generated by the Boral sigma algebra on Rd and the null sigma algebra on Rd.

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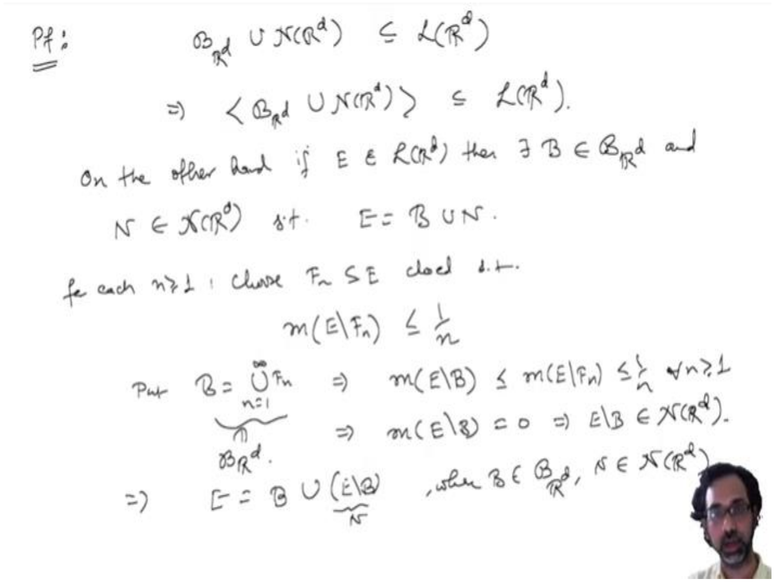

So let us see the proof of this claim first we note that BRd union NRd is a sub collection of Lebesgue measureable sets. Because we know that every Boral subset is Lebesgue measurable because it is a countable union of open sets, or it could be countable union of a countable intersection of open sets and so on. But in any case the open sets are Lebesgue measurable and the null sets null or Boral sets are also Lebesgue measurable.

So we have this inclusion of the union of Boral algebra and the null algebra and inside the Lebesgue sigma algebra. So this implies that the algebra generated by these 2 algebras is sub algebra of the Lebesgue sigma algebra because it is the smallest algebra containing my definition. On the left side we have the smallest algebra containing these 2 collections, on the other hand if E is a Lebesgue measurable subset then I claim that their exist a Boral set B belonging to the Boral sigma algebra and a null set N belonging to the null algebra.

Such that E is a union of these 2 sets B and N so let us see how this is done. So for each N I am going to use the inner approximation by closed sets can choose Fn inside E closed such that measure E – Fn is less than or equal to 1 over n. Now put B to be the union of all these Fn's which implies that the measure of E – B which is less than or equal to E – Fn for all n greater than or equal to 1 which implies that E – B equals to 0.

So E - B belongs to the null algebra on the other hand this is a countable union of closed sets so this belongs to the Boral algebra. So therefore we get that E can be written as union B union E – B and this is our set n that we need. So where B belongs to the Boral algebra and N belongs to null algebra. So we see that the Lebesgue algebra is generated by Boral subsets and null subsets of Rd.