Measure Theory
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Lecture-32
Lebesgue Measurability Under Linear Transformation Construction of Vitali Set-Part 2

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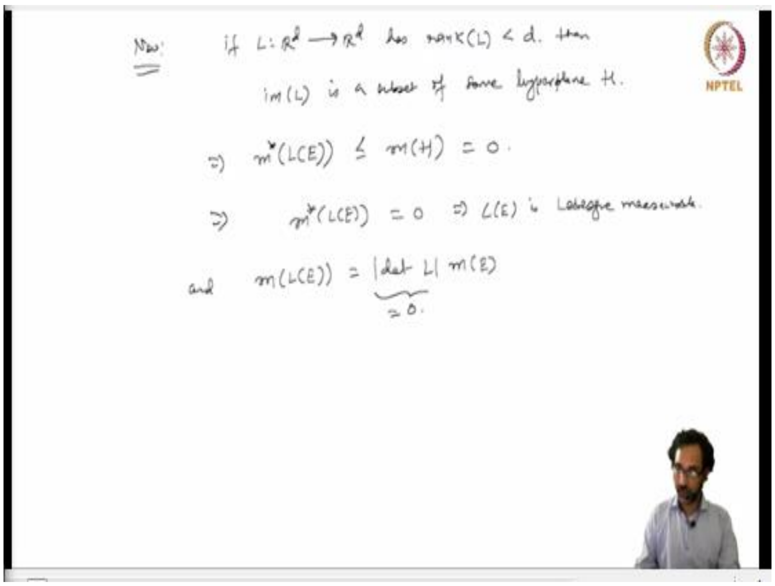

Now we come to the case then.

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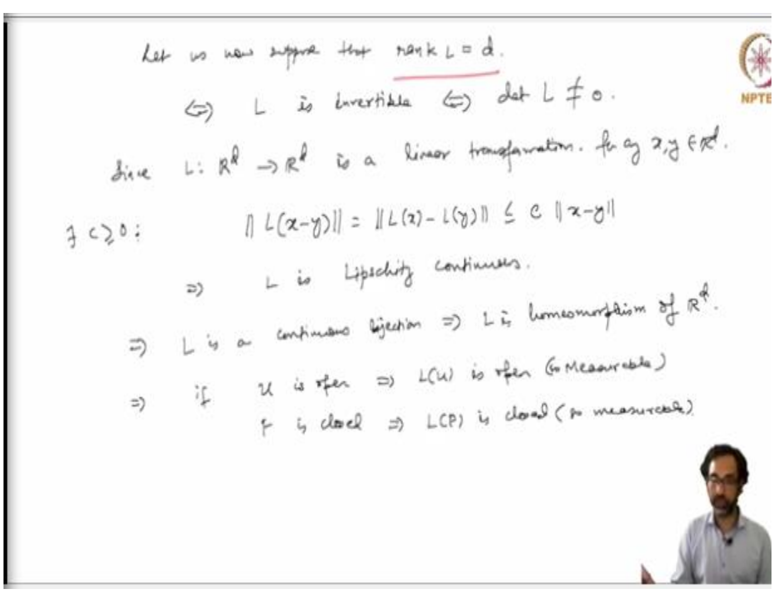

Let us now suppose that rank of L is precisely d, this means that determinant first it means that L is invertible and is also is equivalent to saying that determinant of L is not equal to 0. Now L is a linear transformation since L is a linear transformation for any x, y in R d, we have the following inequality which is that L of x - y equals to L x - L y and this is less than or equal to sum constant times x - y.

So, there exist a positive well non negative constant c such that you have that the lon of L of x - y is less than or equal to this constant c times the lon of x -y which means that L is lipchitz. So, if you recall what is the lipchitz function this is exactly the definition for a lipchitz. And L is lipchitz means also that it is a continuous math. So, L is in our case it is a continuous bijection which implies that L is a homeomorphism of R d.

So, now that we have seen that L is a homeomorphism when rank of L is d then we can say that if u is open. This implies that L u is open and so it is measurable in particular, so measurable and if F is closed this implies that L F is closed and so measurable. So, in the particular case when you have open and close sets we have proved that the image L u and L F are open and closed respectively and therefore both are measurable.

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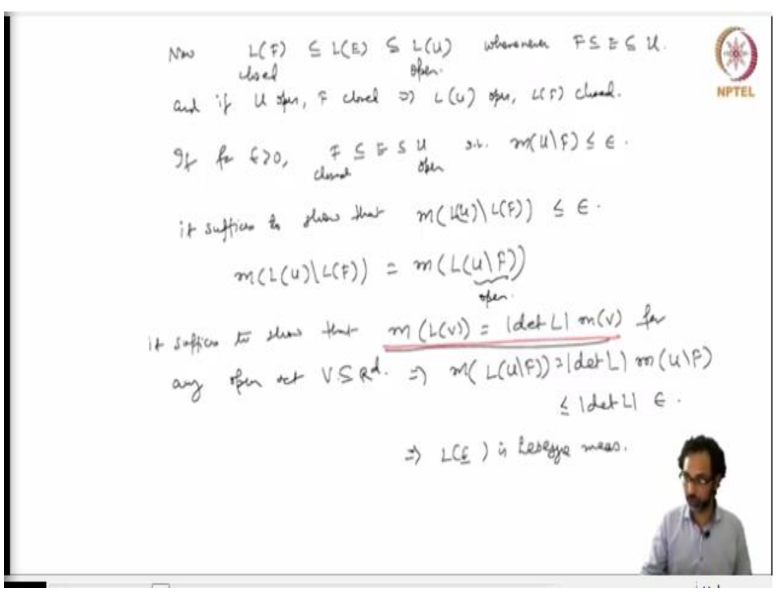

Now observe that L F is a subset L E is a subset of L u whenever F is a subset of E is a subset of u. And if u is open and F is closed then this is closed and this is open, so let me write L u is open and L F is closed as we have see. So, if for epsilon greater than 0, we have a close subset F and an open subset u such that measure of U - F is less than or equal to epsilon.

It is suffices to show that the measure of L u - L F is also less than or equal to epsilon. So, note that m of L u - L F is equal to measure of u of L of u - F because L is invertible. So, it preserves intersections and so therefore we can write this and this is an open set, so it suffices to show that m of L v is precisely the modulus of the determinant of L times m of v for any open set v.

So, this would that the measure of L u - F is less than or equal to or rather is equal to modulus of the determinant of L times the measure of u - F. And this is less than or equal to determinant of L times epsilon and so we would have shown that L E is Lebesgue measurable. So, we will prove that for any open set we have this formula that m of L v is equal to modulus of determinant of L times m of v.

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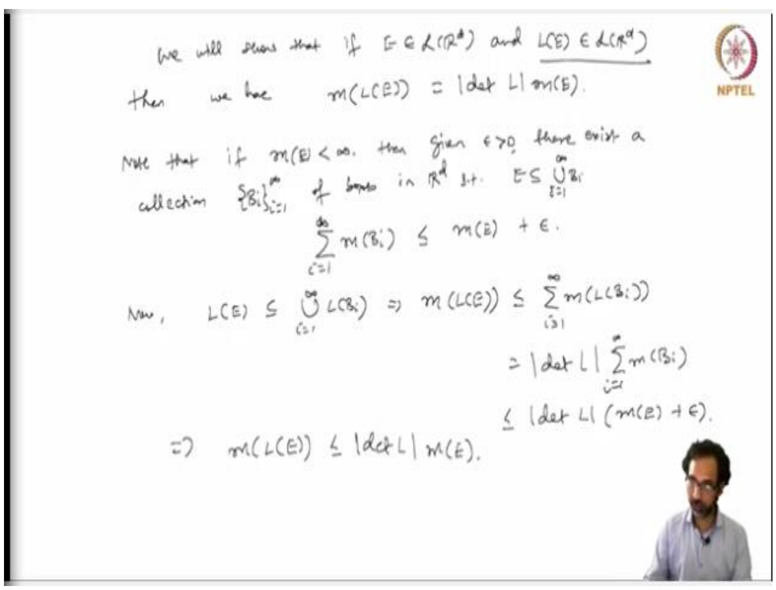

In fact we are going to show something even stronger which is that if we will show that if E belongs to a Lebesgue measurable sets and L E belongs to the Lebesgue measurable set. Then we have measure of L E is equal to determinant of L modulus of the determinant times m E. So, how do we show this, so note that if measure if finite, measure of E is finite then given epsilon greater than 0 there exists a collection B i, i equal to 1 to infinity of boxes in R d.

Such that the sum i equal to 1 to infinity m B i is less than or equal to m E + epsilon. So, this is basically the first definition of the outer measure that we used which use the covering of E with boxes and estimating it by taking an infimum. So, we have this, now L E is then covered, so first of all here E is a covered by these boxes. So, L E is covered by the union of these boxes L B i. So, this implies that the outer measure of L E which is the same as the measure because again L E is assumed to be Lebesgue measurable.

So, this is less than or equal to the sum m L B i, i equal to 1 to infinity. And this we know is equal to modulus of the determinant times m B i for each i, we have this equality which is a measure of L B i is equal to modulus determinant L times m measure of B i. Because we know this for elementary subsets, and so we get modulus of determinant L measure of E + epsilon. So, this implies that measure of L E is less than or equal to modulus of the determinant of L times measure of E.

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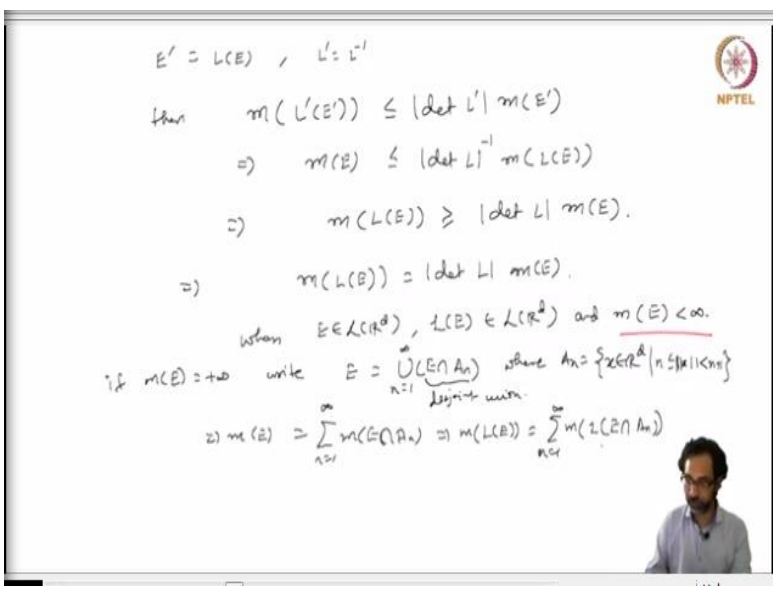

Now I am going to reverse the roles of L E and E, so let E prime L E and L prime B L inverse, then the measure of L prime E prime is less than or equal to determinant of modulus of determinant of L prime times the measure of E prime. But this is nothing but measure of E left hand side is simply E L prime E prime is simply E and this is determinant of L inverse and the right hand side is m of L E.

So, this implies that m of L E is also greater than or equal to determinant of L times measure of E. And so this implies that m L E equals determinant of L modulus times and m E when our assumptions were that first of all that E belongs to L R d, L E belongs to L R d and measure of E is finite. So, if we drop, now we are going to drop this condition that measure of is finite. So, if measure of E is infinite, we can write E as a countable union n equal to 1 to infinity E intersection A n.

Where A n is the set of points in R d says that the norm of x is greater than or equal to n and is strictly less than n + 1. So, this becomes a disjoint union, and so measure of E is equal to the sum of these E intersection A n's, n equal to 1 to infinity which implies that the measure of L E is equal to the sum L E intersection A n.

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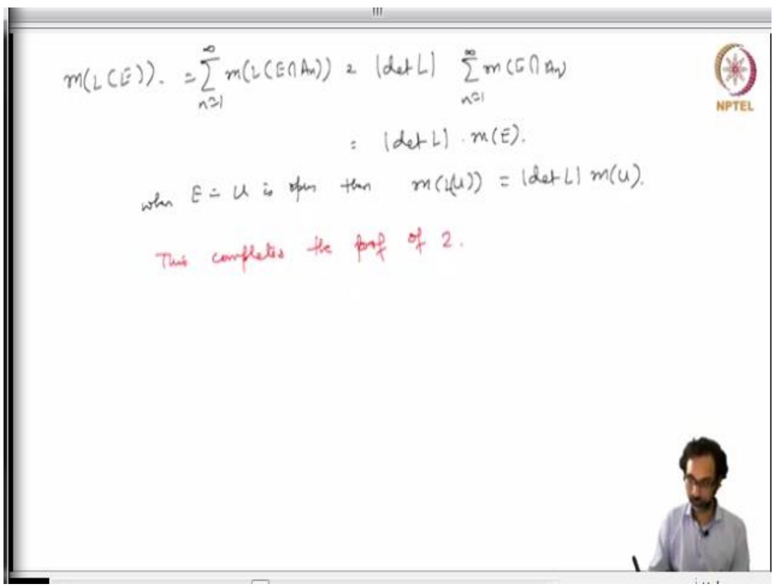

And each of these is now can be written E intersection A n, n equal to 1 to infinity this is modulus of the determinant times m E intersection A n, n equal to 1 to infinity and this is nothing but modulus of the determinant times m E and the left hand side was measure of L E. So, also when measure of E is infinite, we have this equality. So, this proves in particular that when E is an open set, is open then measure of L u is equal to modulus of the determinant of L times measure of u.

And this is what we wanted, because here this is precisely what we wanted to show that for any arbitrary set the measure of L v is equal to determinant of L modulus times m of v which showed that L E is Lebesgue measurable. And now that we know that L E is Lebesgue measurable, then we can again go back to our to something that we just prove that if E is Lebesgue measurable and L E is Lebesgue measurable then we can write this formula. So, basically we are now done with the whole proof, this completes the proof, the proof of the second part.

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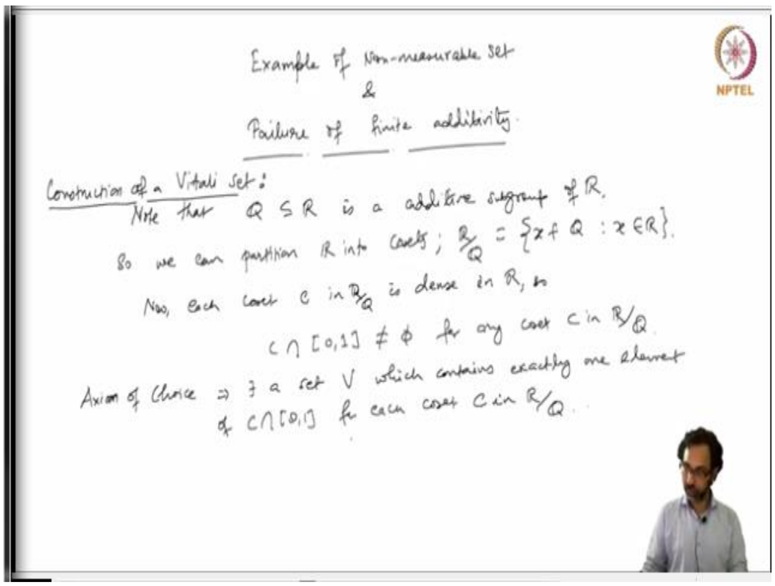

So, we have seen that our Lebesgue measure satisfies our geometric intuitions which is finite, additivity and invariance under translations rotations and reflections. So, now is a good time to see an example of a non measurable set which is not Lebesgue measurable and how this finite additivity property fails for such sets. So, for this, note that, so this is called the construction of a Vitali set.

So, this Vitali set is not going to be Lebesgue measurable. So, first note that the set of rational numbers, this is a additive subgroup of the real numbers, additive subgroup of R. So, we can partition R into co sets which we write is the set of co sets is x + Q, so is that x is in R. So, the quotient group is given by R mod Q and this is simply the co sets right co sets are left co sets of the rationals given by X + Q.

So, now each co set c in R mod Q is dense in R, so c intersection the interval 0, 1 is non empty for any co set, c in R mod Q. So, now I am going to use the axiom of choice again to construct this Vitali set. So, axiom of choice implies that there exists a set which I call V which contains exactly one element of c intersection 0, 1 for each co set c in R mod Q. So, we have constructed a set which for each co set c picks out a number in c intersection 0, 1 and this V is precisely made up of such elements picking up one from each co set c which lies in the interval 0, 1.

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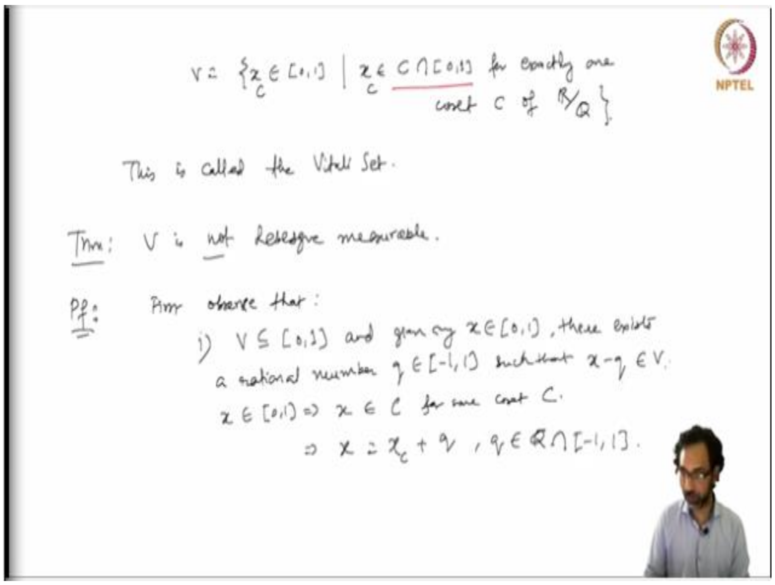

So, V is the set of numbers x in the interval 0, 1 such that x belongs to c intersection 0, 1 for exactly one co set c of R mod Q. So, this is our this is called Vitali set and the theorem is that V is not Lebesgue measurable. So, let us try to prove this, so first known that, first observed that we can write two things, first is that V is a subset of 0, 1 by construction and given any x in 0, 1.

There exists rational number Q in same - 1, 1 such that x - q belongs to V. So, this is by the construction of the Vitali set, because we are choosing only one number from each co set c intersection 0, 1. So, for any given x, there exists a rational number q such that x - q belongs to V. This is because x belongs to 0, 1 implies that x belongs to c for some co set c. Because this is a partitioning of R into co sets, so in particular for x a real number between 0 and 1, it belongs to some co set it must belong to some co set c.

And choosing the representative from the Vitali set is implies that x belongs to. So, let me here write x c for the representative inside the Vitali set for the co set c. So, this means that x equals x c plus some rational number Q and because both x and x c belong to 0, 1 this is in fact a rational in - 1, 1. So, this is why we get a rational number q such that x - q belongs to V.

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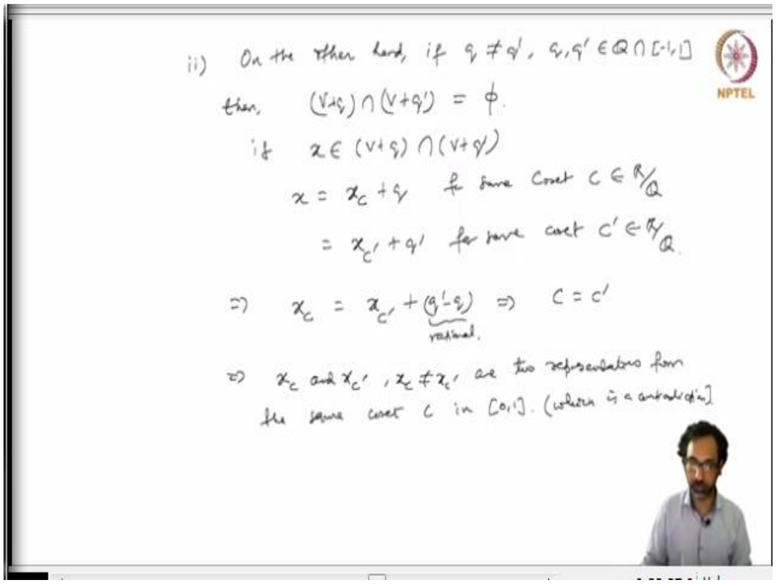

On the other hand, if q is not equal to q prime for rationals q and q prime in - 1, 1. Then V + q intersection V + q prime is empty, these 2 are disjoint subsets of - 1, 1. So, this is because if x belongs to v + q intersection v + q prime. So, x can be written as x c + q for some co set c in R mod Q. And it is also equal to x c prime + q prime for some co set c prime in R mod Q. So, this implies that x c is equal to x c prime + q prime - q but this is again a rational number.

And which means that c must be equal to c prime and this means that x c and x c prime are two representatives from the same co set which is not, this implies that x c and x c prime with x c not equal to x c prime are now two representatives from the same co set c belonging to 0, 1. So, x c and x c prime both belong to 0, 1 and this is not which is a contradiction. So, we see that we have whenever q is not equal to q prime these two translates v + q and v + q prime are disjoint.

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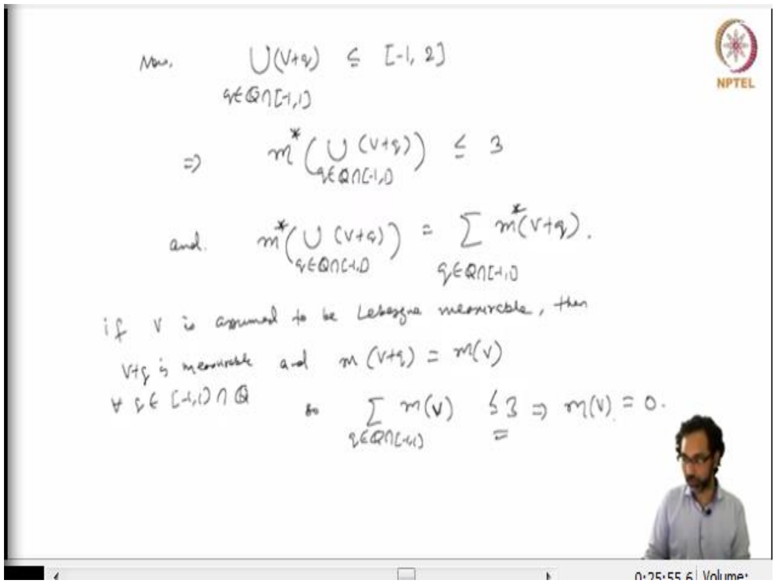

Now, note that the union V + q when q ranges in the rationals between - 1 and 1, this is a subset of - 1, 2 the interval - 1, 2. So, this implies that the measure of this union, this is a countable union because it is a union over rationals in - 1, 1. So, this measure is bonded above by 3 and also we have that this is a countable union of disjoint sets. So, we have this sum over the rationals in - 1, 1 m V + q.

So, I should rather write m star instead of m because we still do not know whether any of these subsets are Lebesgue measurable. So, if V is assumed to be Lebesgue miserable. Then, of course m star v + q, so v + q measurable v + q is measurable for all q in - 1, 1 intersection Q. And the measure of V + q now I am removing star because it is a Lebesgue measure is the same as the measure of V.

So, we have that the sum - 1, 1 m v is less than or equal to 3 which implies that the measure of V must be equal to 0 because if it is positive then it is going to be infinite. But here we have a bound of 3 on this whole thing on this infinite sum, so m v must be 0.

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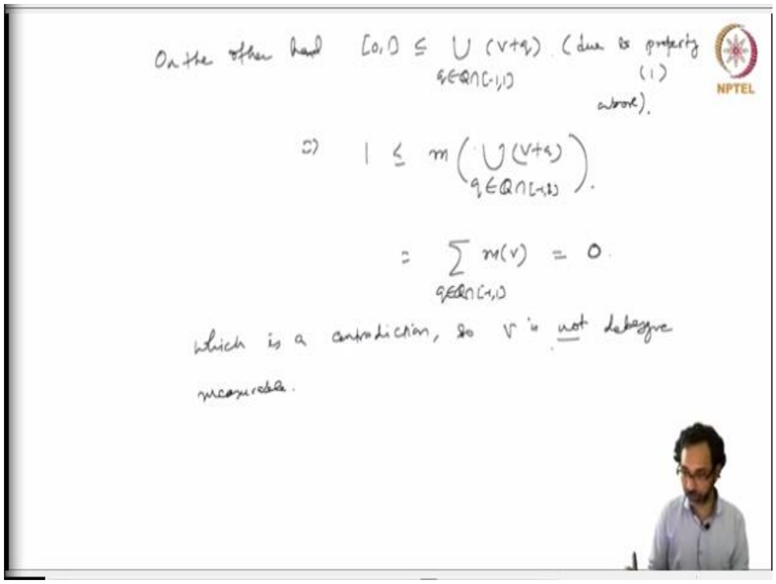

On the other hand we have that 0, 1 is contained inside this union q in - 1, 1 v + q. This is because of due to property 1 above and this is a disjoint collection due to property to above. So, this means that, so 1 is less than or equal to the measure of this set v plus this is the union of v + q's. On the other hand this was the sum of these things measure of v + q but it was v was assumed to be Lebesgue measurable.

So, it is measure of v and this is 0 because if v was assumed to be living measurable, then m v must be 0. So, this is a contradiction which is a contradiction, so, v is not Lebesgue measurable. So, this is the construction of the Vitali non Lebesgue measurable set. And now I am going to show that this v fails the finite additivity property.

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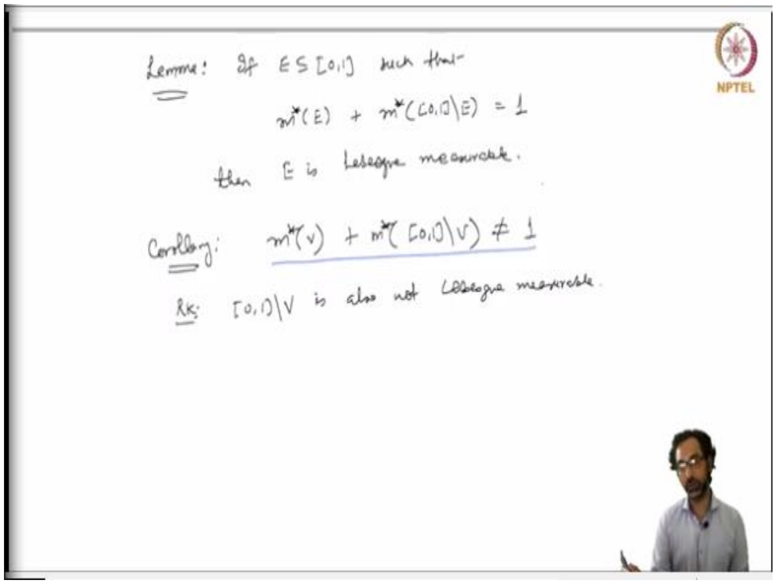

So, we will prove that this following lemma that if E is a subset of R d rather 0, 1. Such that the outer measure of E + the outer measure of 0, 1 - E is equal to 1 then E is Lebesgue measurable. So, assuming that this lemma holds as an immediate corollary of this lemma we have that for the Vitali set 0, 1 - v is not equal to 1, just as a remark, note that this compliment 0, 1 - v is also not Lebesgue measurable.

Because Lebesgue measurability is closed under compliments. So, if this was Lebesgue measurable, if you take the compliment, then you will get v which would then be Lebesgue miserable and so we have a contradiction. So, if v is not Lebesgue measurable, the complement inside 0, 1 of v is also not Lebesgue measurable.

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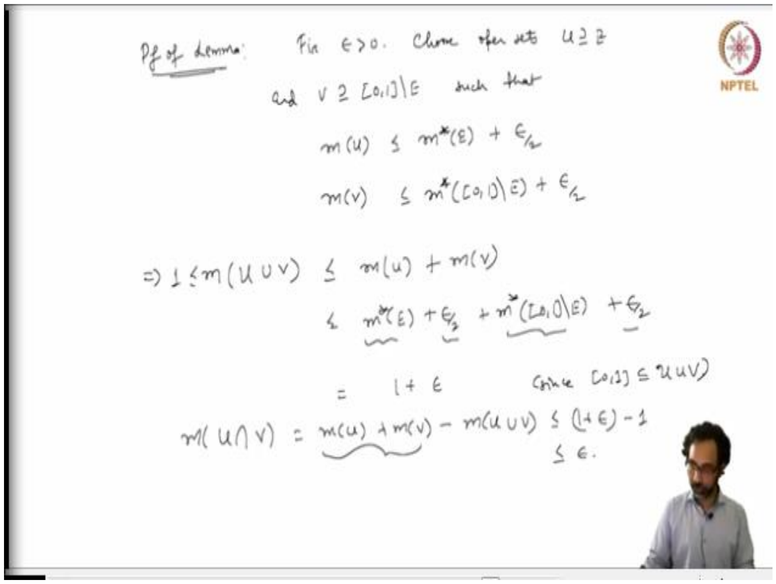

So, let us now prove the lemma, so proof of lemma, so let us fix epsilon greater than 0. And we choose open sets u which cover E and V which covers 0, 1 - E. Such that m star u rather m u is less than or equal to m star E + epsilon and m v is less than or equal to m star 0, 1 - E + epsilon. So, let me take epsilon by 2, so that our analysis will be a little bit easier later on. So, this implies that the measure of the union of these two sets u and v is less than or equal to m u + m v which is less than or equal to m star E + epsilon by 2 + m star 0, 1 - E + epsilon by 2.

But these 2 gave you the sum 1 and these two gave you the sum epsilon, so in total it is 1 + epsilon. So, the measure of the union of u and v is bounded above by 1 + epsilon. On the other hand, the measure of u intersection v which is equal to m u + m v - m u union v and this is less than or equal to 1 + epsilon. So, this first term is less than or equal to 1 + epsilon and the second term is greater than or equal to 1.

Because since note that since 0, 1 is a subset of u union v. So, the measure of u union v is in fact greater than or equal to 1, so this is less than or equal to epsilon, so the intersection has upper bound epsilon.

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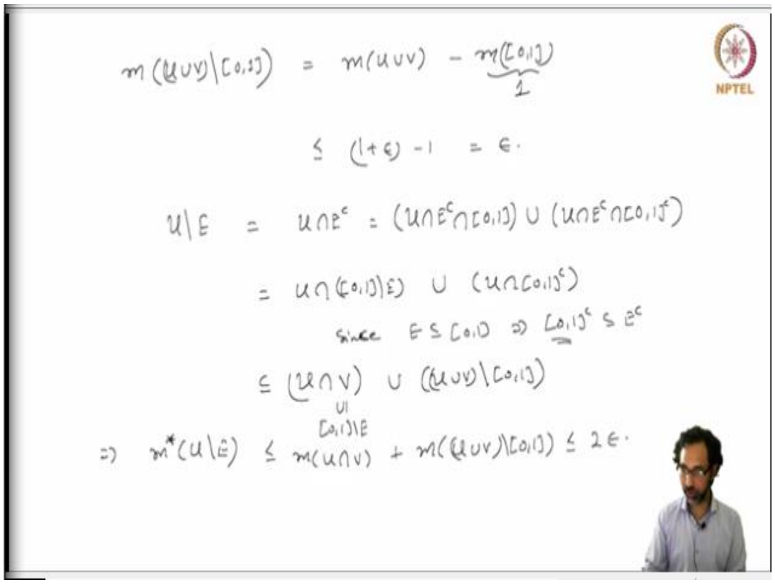

In turn this means that the measure of u union v minus this interval 0, 1 which is equal to measure of union of u and v - measure of 0, 1. This is a force 1, and this was bounded above by 1 + epsilon - 1, so this is again bounded above by epsilon. So, now if you take u - E then this is let me first write that u - E. So, you can write this as u intersection E complement which can be further written as u intersection E complement intersection 0, 1 union with u intersection E complement intersection 0, 1 compliment.

So, the first one this is nothing but u intersection 0, 1 - E and the second one is nothing but u intersection 0, 1 compliment. This is because since E is a subset of 0, 1, so 0, 1 compliment is a subset of E compliment and so the intersection is simply the smaller set 0, 1 compliment. So, now this is a subset of u intersection v because v covered v is an open set that covered 0, 1 - E. And the second one is a subset of u union v - 0, 1.

Now, we have control over the measures of both the sets. So, we can immediately get back the measure of u - E is less than or equal to the measure of u intersection v plus the measure of u union v - 0, 1. So, both are equal to less than or equal to epsilon, so this is equal to 2 epsilon and so E is Lebesgue measurable. So, this finishes the proof that finite additivity property does not hold for this Vitali set.

Because if there was equality then we should have been Lebesgue measurable which we have seen that it is not the case. So, we have seen that for non measurable sets finite additivity fails to hold.