Measure Theory
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Lecture-30
Lebesgue Measurability Under Linear Transformation Construction of Vital Set-Part 1

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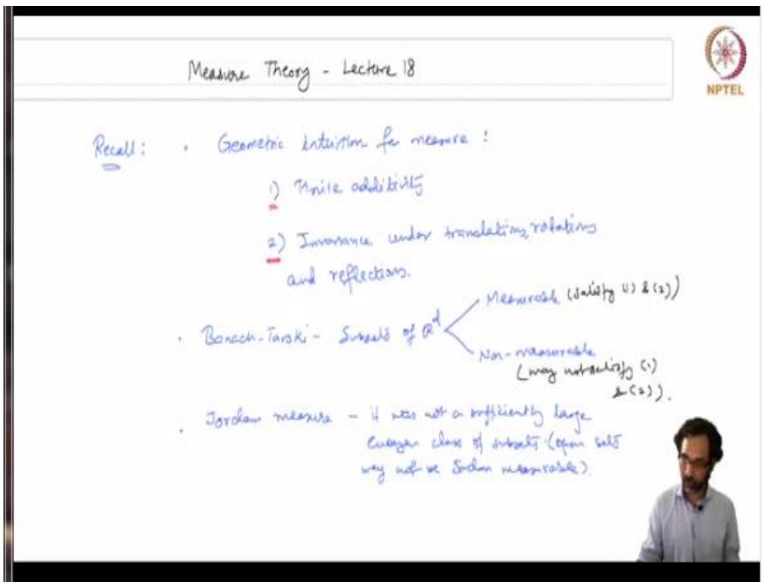

Now that we have seen so many properties of the Lebesgue measure and Lebesgue measurable sets. It is a good time to look back at what we are trying to do and recall where we started off from. So, let me recall our original goals, so we were guided by some geometric intuition which our measures were expected to satisfy. The first one was finite additivity property and the second one was invariance under translations, rotations and reflections.

So, we had these two main geometric intuitions in mind when we were trying to define what should be a concept of measure for arbitrary subsets of R d. But the Banach Tarski paradox forced us to look at to separate the classes of subsets of R d into two classes subsets of R d were divided into two classes measurable and non measurable. Because the Banach Tarski paradox establishes the existence via the axiom of choice.

The existence of subsets of R d which failed to have finite additivity if 2 is assume then one fails. So, we have to divide the subsets of R d into 2 classes, the measurable ones were supposed to satisfy 1 and 2 and the non measurable ones may not satisfy 1 and 2. So, we started off with the concept of a Jordan measure which was defined via the volume of boxes.

And then we gradually increased our classes of subsets of R d into in consideration from boxes to finite union of boxes and then to Jordan measurable sets. But we still felt that even though the Jordan measure respected finite additivity and invariance under translations, rotations and reflections for Jordan measurable sets. But still it was not closed under countable unions and countable intersections and left out many important subsets of R d such as open sets which were not Jordan measurable, so we went further.

So, the drawback for Jordan measure and Jordan measurable subsets was that, it was not as sufficiently large enough class of subsets. So, in particular open sets may not be Jordan measurable. So, since we wanted to have this connection with the topology underlying topology of the Euclidean space R d. We wanted to have at least the open subsets of R d should be measurable in suitable notion of measure.

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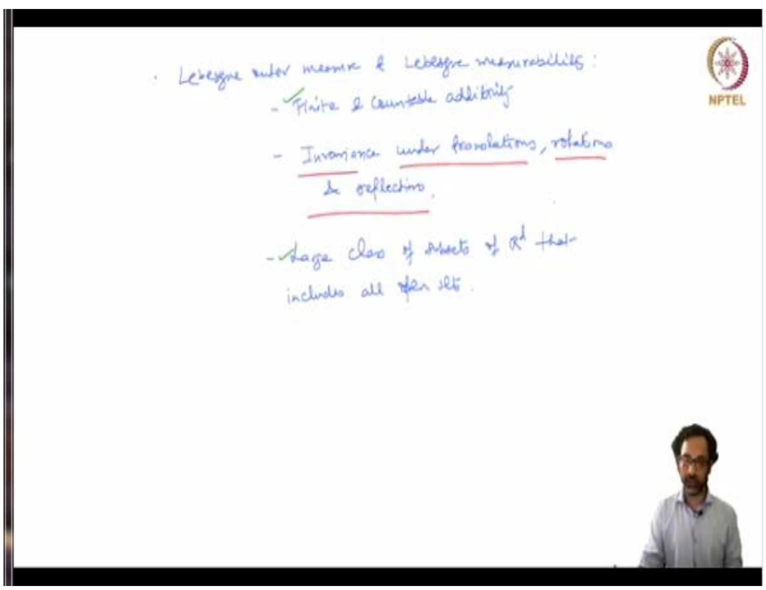

So, this is why we define the notion of a Lebesgue outer measure and Lebesgue measurability. And we proved various properties such as finite and countable additivity for Lebesgue measurable sets. And we still have not proved though that Lebesgue measurable sets is invariant under translations, rotations and reflections, so we have proved this. And of course this is a large class of subsets of R d that includes all open sets.

So, this we have also seen but we have not seen yet, the second property which is invariance under translations, rotations and reflections. And, so this was one of our main geometric intuitions for the properties of the measures that we wanted. So, in this lecture, we will see this property of invariance of the Lebesgue measure under translations, rotations and reflections.

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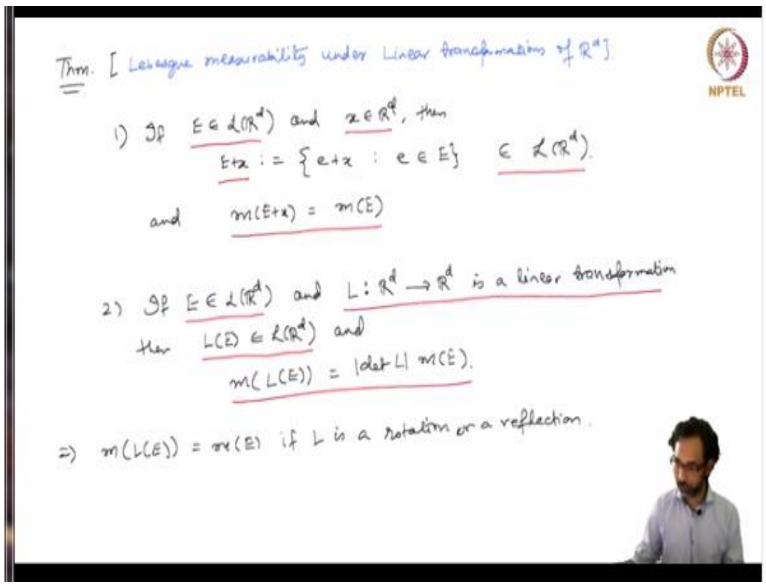

So, this is the theorem that we want to prove which is the Lebesgue measurability under linear transformations of R d. This kind of result we have already seen for the Jordan measure and this theorem is for the Lebesgue measure. So, the first part says that if E is Lebesgue measurable and x is a vector in R d then the translated set E + x is also Lebesgue measurable and the Lebesgue measure of the translated set is the same as the Lebesgue measure of the original set.

And the second part says that if E is Lebesgue measurable and L is a linear transformation of R d. Then the image of E under L is a Lebesgue measurable set and the measure transforms as we would expect which is a multiple of the measure of E and the multiple is given by the modulus of the determinant of L. So, in this way this is an extension of the transformation for Jordan measurable sets.

And this shows also that from the second part that this is m of L E is equal to m of E if L is a rotation translation we have already seen is a rotation or a reflection. Because if L is a rotation or a reflection then it is an orthogonal matrix and an orthogonal matrix has determinant 1. So, we have m L E equals m E of course the first part is translation invariance and the second part is rotation invariance and reflection invariance.

So, in this way we would have prove the geometric intuition that we started off with which is the measure should satisfy missing variance under translations, rotations and reflections and it should satisfy finite sub additivities also. Since we have already seen countable additivity we have only have to prove this invariance under translations, rotations and reflections.

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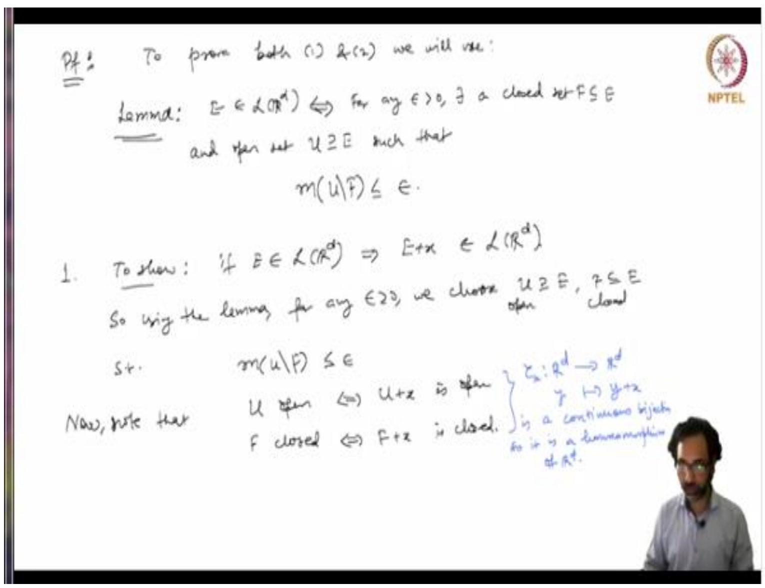

So, let us see how this is proved. So, to prove this, so let us write first what we are going to use to prove this. To prove both 1 and 2, we will use the following lemma which gives an another equivalent characterization for Lebesgue measurability which is that E is Lebesgue measurable subset of R d. If and only if for any epsilon greater than 0 there exists a closed subset, close set F and an open set u which contains E such that the measure of u - F is less than or equal to epsilon.

So, one can just use the inner approximation by close sets and outer approximation by open sets to prove this equivalence that E is Lebesgue measurable if and only if this condition holds. So, let us see how we use this lemma to prove the first part, so first we will show that to show that if

E is Lebesgue measurable then E + x is also Lebesgue measurable. So, using the lemma for any epsilon greater than 0 we choose u which contains E and is open.

And F which is a subset of E and is closed such that the measure of u - F is less than or equal to epsilon. Now note that u open is equivalent to saying that u + x is open and u closed or rather F closed is equivalent to saying that F + x is closed. This is because the reason is that the translation map tau x from R d to R d which takes y to y + x is a continuous bijection, so it is a homeomorphism of R d.

So, this means that it takes open sets to open sets and close sets to close sets and so we have that if u is open it is equivalent to saying that u + x is open. And if F is closed then F + x is closed and vice versa.

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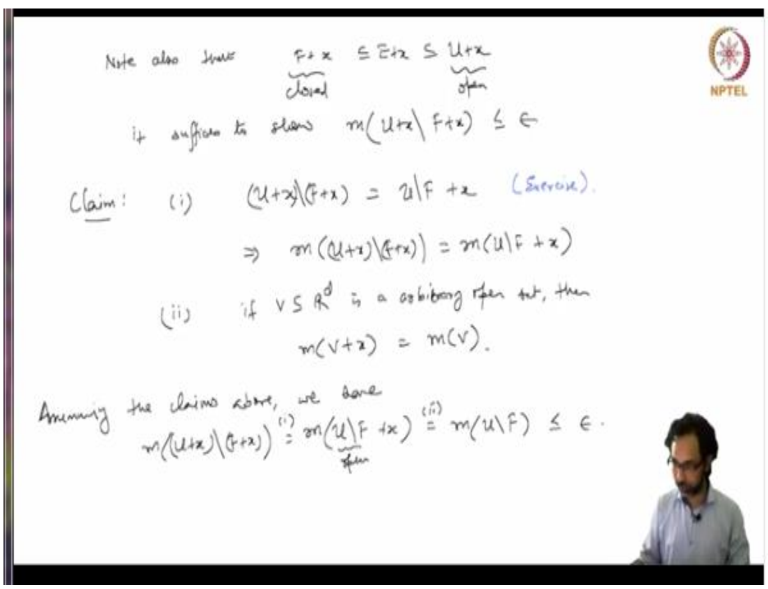

Note also that F + x is a subset of E + x is a subset of u + x and this is an open set and this is a closed set. So, it is suffices to show that the measure of u + x - F + x is less than or equal to epsilon. So, for this I claim two things first is that the measure of u + x well actually u + x - F + x = u - F + x which implies that the measure of u + x - F + x is equal to the measure of u - F + x.

And the second is that if v is an well arbitrary open subset then m of v + x is equal to m of v. So, with these two things we can immediately conclude, so assuming the claims above we have the measure of u + x - F + x is equal to the measure of u - F + x. And now note that this is an open set because u is open and the complement of F which is closed is open.

So, it is an intersection of 2 open sets therefore it is open. So, by 2, so this is why first part, the second part then shows that this is simply m of u - F and this is less than or equal to epsilon, so we are done. So, we just have to show the claims, so the first part is very easy is just a set theoretic argument, so I leave it as an exercise, so I will prove only the second part.

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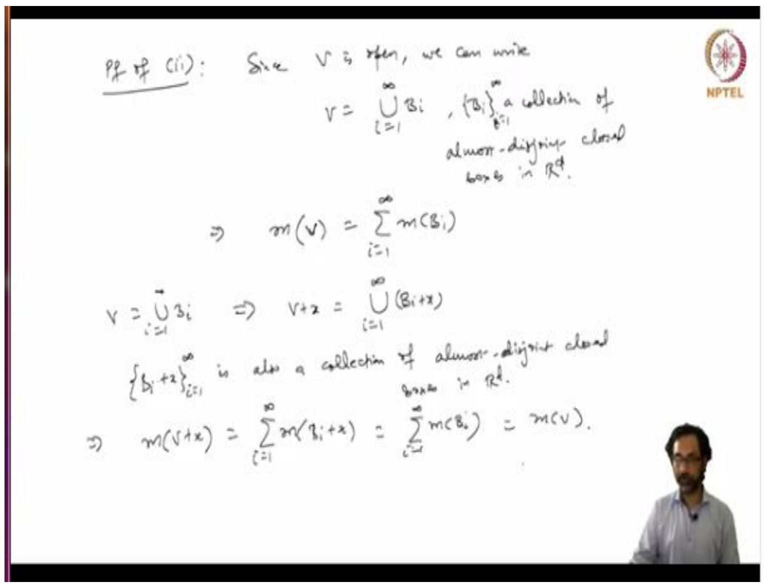

So, proof of two, so since V is open it is in we can write V as a countable union of boxes B i collection. So, B i n equal to i equal to 1 to infinity a collection of almost disjoint closed boxes in R d. So, we have shown this, we have used this lemma before, I did not prove it but I referred to (()) (16:18) book for a proof that any open set is a countable union of almost disjoint closed a boxes in fact cubes in R d.

So, this implies that m of V is the sum of m B i, i equal to 1 to infinity. Now note that if V can be written as the union of B i's then V is the union of these B i’s, this implies that V + x is then the union of this sets B i + x, i equal to 1 to infinity. And note that B i + x i equal to 1 to infinity is also a collection of almost disjoint closed boxes in R d. And we already know that the translation invariance holds for boxes. So, we have the measure of m of V + x is equal to the sum B i + x i equal to 1 to infinity.

This is nothing but i equal to 1 to infinity m B i because B i is a elementary and we have shown that translation invariance holds for elementary sets, even Jordan measurable sets. So, we get m v on the right hand side, so m v + x is equal to m v for any open set v. So, this proves that our second claim and since I have already left the first is an exercise then it shows that E + x is Lebesgue measurable.

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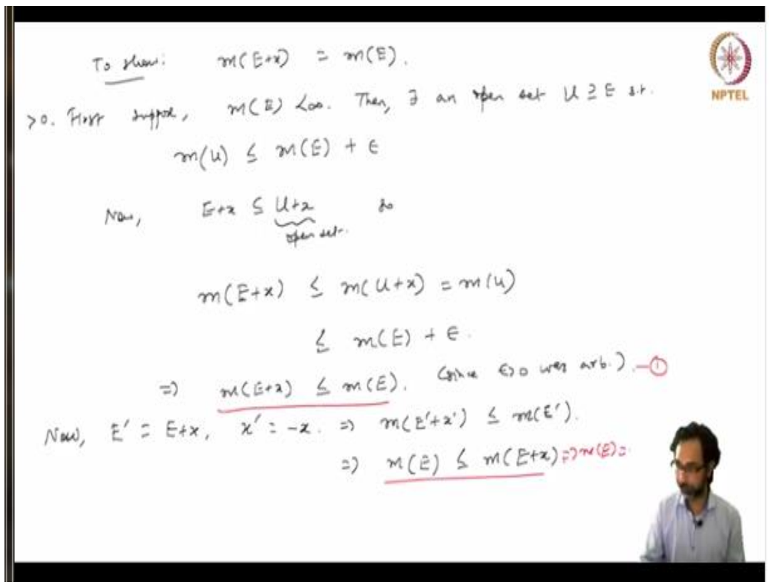

Now we still have to show that the measure of E + x is equal to the measure of E. So, first suppose that the measure of E is finite ok, then there exists an open set u containing E such that measure of u is less than or equal to measure of E + epsilon. So, I am going to fix epsilon greater than 0 and then if measure of E is finite then you can find an open set containing E such that the measure of u is bounded above by measure of E + epsilon.

Now E + x is contained in u + x and this is an open set. So, the measure of E + x, so I am writing measure because we have already shown that I am writing m instead of m star because I have already shown that E + x is Lebesgue measurable. So, this is less than or equal to m u + x by monotonicity. And this is equal to m u by the translation invariance for open subsets that we have proved and this is less than or equal to m E + epsilon.

For any arbitrary epsilon this implies that m E + x is less than or equal to m E since epsilon was arbitrary. Now this holds for any Lebesgue measurable set E and any vector x, so I am going to put E prime as E + x and x prime as - x. So, this implies that measure of E prime + x prime is less than or equal to measure of E prime by what we have just shown. But E prime + x prime is nothing but E because it is E + x - x.

So, the left hand side is just E and the right hand side is E + x. So, we have shown that m E + x is less than or equal to m E. So, this is the first inequality and the second inequality is the reverse one. So, this concludes the proof that m E equals, so let me write it here and m E equals m E + x.

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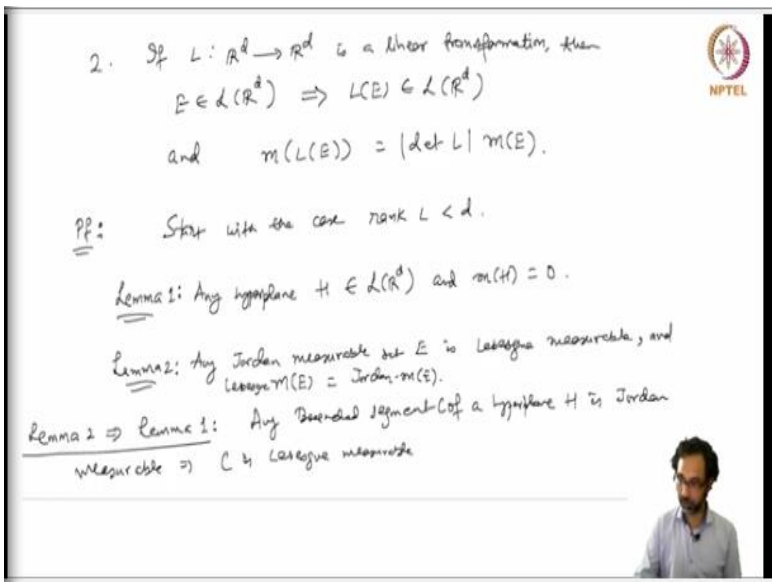

Now, let's come to the second part, part 2 which is a more interesting if you want. That if L is a linear transformation then E being Lebesgue measurable implies that L E is Lebesgue measurable belongs to L R d. And the measure of L E is precisely modulus of the determinant times m E where the equality holds in the extended real numbers, so both sides can be + infinity as well nevertheless this equality holds.

So, let us see the proof for this statement, so we have to start with the case when the rank of L is strictly less than the dimension d on which it is defined. So, we expect that the image will lie in a hyper plane of strictly of dimension strictly less than d. And so it should have measures 0, just as in the case of the Jordan measure. But for this to be proven we need the following lemma to show that any hyper plane H belongs to the class of Lebesgue measureable sets and the measures is 0.

So, we already know that bounded segments of hyper planes are limited or Jordon measurable and they have Jordan measure is 0. So, if we can write the hyper plane as a countable union of disjoint bounded segments of hyper planes. Then the resulting thing will be Lebesgue measurable and the measure will still be 0 because each component will have measures 0. So, but we still need to prove that if a bounded segment is Jordan measurable, then it is Lebesgue measurable because the 2 things are have different definitions.

So, we still have not shown this lemma that this is another lemma interesting in it is own right that every any Jordan measurable subset Jordan measurable set E is Lebesgue measurable. So, let me write this is lemma 1 and this is lemma 2, so first let us see how lemma 2 implies lemma 1. So, this is because any bounded segment, bounded segment mean a subset of the hyper plane which is bounded.

So, I am calling this a bounded segment of a hyper plane H is Jordan measurable, so this we have proved before. So, this implies that let me denote the bounded segment as c, this implies that c is Lebesgue measurable.

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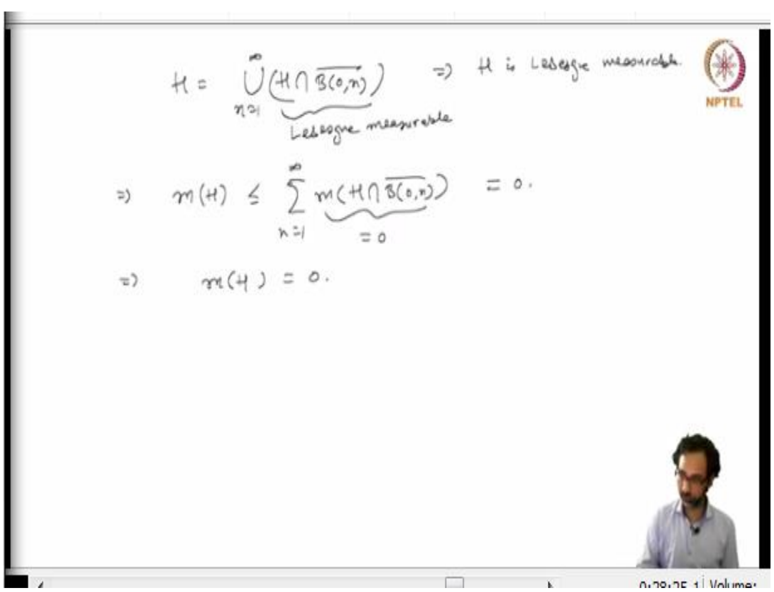

And then we can write H as the union of H intersection the closed ball of with center 0 and radius n, n equal to 1 to infinity. So, this is now a countable each one is Lebesgue measurable because it is Jordan measurable and so this implies that H is Lebesgue measurable. And so, the measure of H is bounded above by the measure of these bounded segments n equal to 1 to infinity.

And because it is Jordan measurable, we have seen that the Lebesgue outer measure coincides with the Jordan measure for Lebesgue measurable sets. So, it implies that this is 0 for each end and so this is 0 and therefore the measure of H itself is 0. So, lemma 2 implies lemma 1, so let us try to show that any Jordan measurable set is Lebesgue measurable and I should add here and the Lebesgue measure m E is equal to the, so let me write Lebesgue.

Because we are using for both notations we are using m, so on the left hand side we have the Lebesgue measure which is by definition. The restriction of the Lebesgue outer measure to Lebesgue measurable sets and on the right hand side we have the Jordan measure. So, we have shown that the Lebesgue outer measure coincides with the Jordan measure for Jordan measurable sets.

So, any Jordan measurable set is Lebesgue measurable and of course the measures do coincide. So, let us see how to prove lemma 2.

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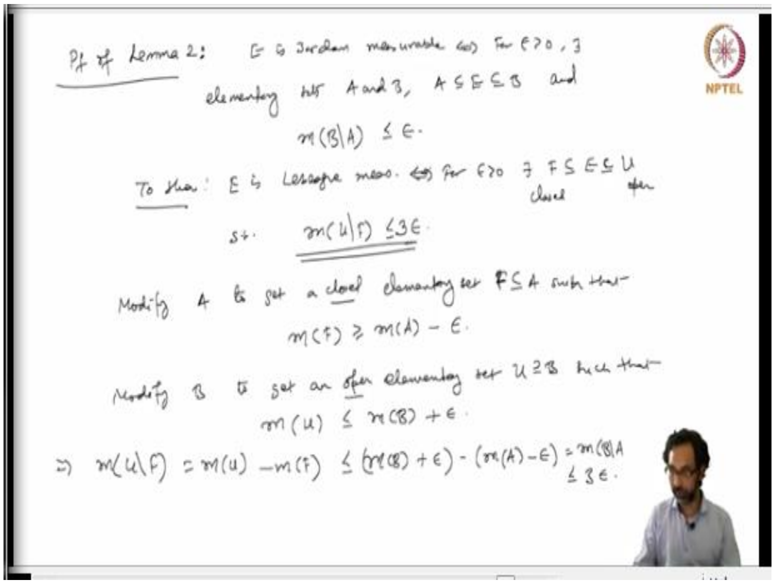

So, proof of lemma 2 which is quite easy, remember that E is Jordan measurable has the following equivalent characterization for epsilon greater than 0. There exists elementary sets A and B such that e is a subset of E is a subset of B and the elementary measure of B - A is less than or equal to epsilon. So, now we are going to show that E is Lebesgue measurable to show E is Lebesgue measurable.

So, we will again use the characterization that we have just used which is that for any epsilon greater than 0, there exists a closed subset F of E and an open subset u of E such that the measure of u - F is less than or equal to epsilon. So, I am going to produce u and F by modifying A and B. So, modify A to get a closed elementary set F such that the measure of F is greater than or equal to the measure of A - epsilon.

So, when we were trying to show a countable additivity, we have used this fact that we can have control over the volume of the elementary sets. So, that we can get closed and open sets respectively. So, here we have taken a close elementary subset F of A such that this is true and then modify B to get an open elementary subset u containing B such that the measure of u is less than or equal to measure of B + epsilon.

So, note that all these measures are that we have here are finite because E is Jordan measurable, so by definition it is bounded and so all these measures are in fact finite. And so here we can immediately get that m of u - F is equal to m u - m F because this is finite things and this is less than or equal to m B + epsilon - m A - epsilon which is equal to m of B - A + 2 epsilon and this is less than or equal to 3 epsilon.

So, in fact we have shown that this is less than or equal to 3 epsilon in the criteria for Lebesgue measurability which is still fine because it is still an arbitrary number 3 epsilon, so E is Lebesgue measurable.

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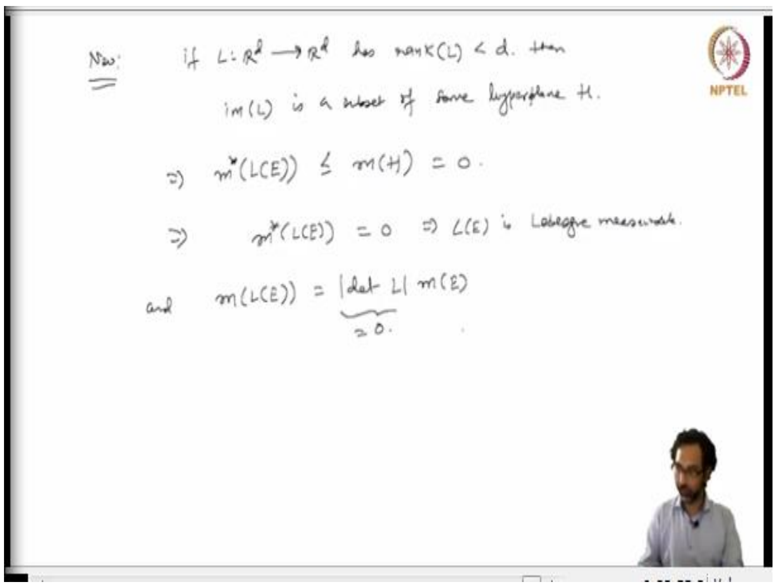

So, now going back to the linear transformation L, if L has rank strictly less than d then the image of L is a subset of some hyper plane H which implies that L E. The outer measure of L E is less than or equal to the outer measure which is this Lebesgue measure of the hyper plane H and this is 0 as we have shown. So, this implies that the outer measure of L E is in fact 0 which implies that L E is Lebesgue measurable.

As well as the fact that m of L E is equal to modulus of the determinant of L times m E because this is precisely 0. So, for the case when the rank is strictly less than d, this is fine.