MEASURE THEORY

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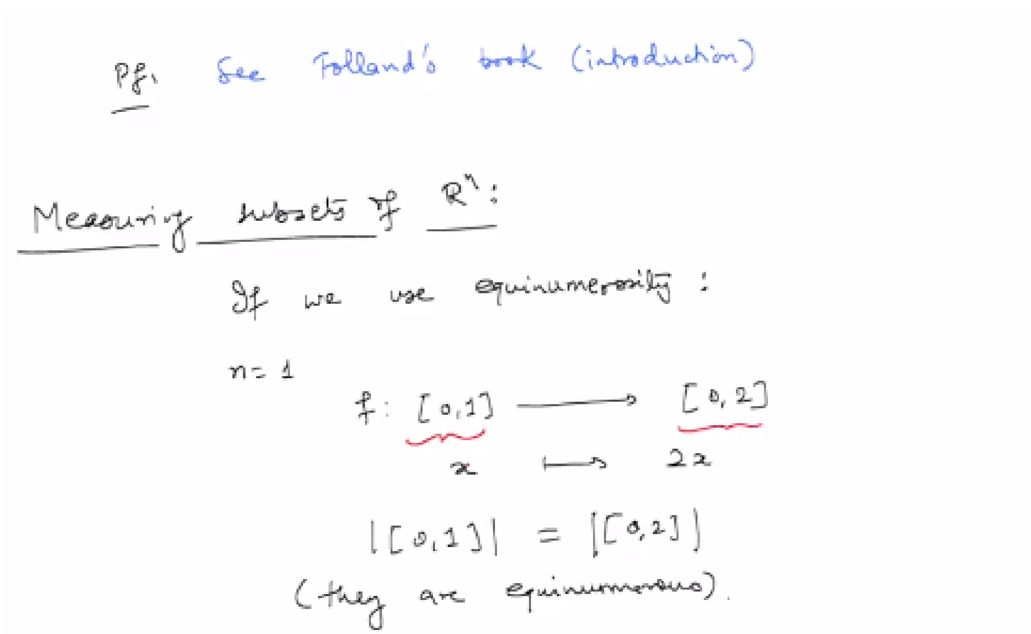
Lecture-3
Infinite Sets and the Banach-Tarski Paradox - Part 2

We have seen that we can measure relatively the size of infinite sets. Now we come back to the question of measuring subsets of $\mathbb{R}^n$ and we will see that the problem of measure starts here. Now, whenever we are trying to measure subsets of $\mathbb{R}^n$, they can be finite sets or infinite sets. If you simply use the cardinality to be infinity, then we only get infinite values and we cannot do any arithmetic with it, so it is not very useful.



So, if we use the concept of equinumerosity, then one can still have an idea of the relative size of subsets of $\mathbb{R}^n$. Suppose that $n = 1$, and we take two sets $[0, 1]$ and $[0, 2]$. Now there is an injection $f : [0, 1] \to [0, 2]$, but one can easily also produce a bijection simply by taking $x \mapsto 2x$. We see that these two subsets of $\mathbb{R}$ are equinumerous,

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but our geometrical intuition says that the set $[0, 1]$ is half the size of the set $[0, 2]$ in terms of length.

So, if we want to have our geometrical intuition conform with the idea of equinumerosity, then it is no good because it says that both these sets are of the same size. (Refer Slide Time: 03:07)

In fact, one can even have for any $\epsilon > 0$, the set $[0, \epsilon]$ and the set $[0, 2]$ are equinumerous. So, you can take ϵ to be as small as possible and still they are equinumerous. So, it not only violates our geometric intuition, but it gives you a sort of contradictory intuition that as you can have as small a set as possible $[0, \epsilon]$, but it will still be of the same size of $[0, 2]$. This violates our geometric intuition.

Thus we are trying to search for a way of assigning numerical values to subsets of $\mathbb{R}^n$ which will conform to the geometrical intuition that we have in terms of length, area or volume, etc. But such a thing is not possible was shown by a theorem of Banach and Tarski. This is the famous Banach-Tarski paradox and this states that:

Theorem 0.1 (Banach-Tarski Paradox)**.** *There exists a decomposition of the* 2*-spheres into finitely many pieces, which can be rearranged to form* 2 *spheres of the same size as the original one.*

In other word, we can break a part something as a size of peanut or a pea, and we can reassemble it into many many pieces of the same size. And we can repeat this over and over again, and we can produce by taking the rearrangements to produce a something of the size of the whole earth. This is a kind of a paradox. But this paradox uses the axiom of choice in a fundamental way. (Refer Slide Time: 06: 13)

Axiom of choice: Given a collection of non-empty disjoint sets $\mathcal{A}$, $\exists$ a set C such that $C \subseteq \bigcup_{A \in \mathcal{A}} A$ and $|C \cap A| = 1$ for each $A \in \mathcal{A}$.

Thm: A is an infinite set $\iff$ there is a bijection between A and some proper subset of itself.

Banach-Tarski paradox violates the principle of finite additivity.

So, what is the axiom of choice? Let me recall the axiom of choice. ***Axiom of choice***: Given a collection of non-empty disjoint sets $\mathcal{A}$, $\exists$ a set C such that

$$C \subseteq \bigcup_{A \in \mathcal{A}} A \quad \text{and} \quad |C \cap A| = 1, \text{ for each } A \in \mathcal{A}.$$

One can think of it as choosing one element from each of these constituent sets A, and we can form a new set C by taking arbitrarily many such choices, and this is why it is called the axiom of choice.

A consequence of the axiom of choice for infinite sets, we have seen that if there is a bijection between a set and a proper subset of itself, then it is infinite. A consequence of the axiom of choice is given by the following theorem:

Theorem 0.2. *A is an infinite set if and only if there is a bijection between A and some proper subset of itself.*

This characterizes the infinite set as the bijective correspondence between A and some proper subset of itself.

Now we are coming back to the Banach-Tarski paradox. It says that you can decompose the unit sphere into finitely many pieces and reassemble them to give 2 copies of the same original sphere. This violates our principle of cardinality. So, the Banach-Tarski paradox violates the principle of finite additivity. It is something like saying that: You have a cup of water, and you can break it. You can divide this cup of water into many pieces and reassemble it, and you will get a whole gallon of water. That is not possible and it leads to this paradox. This is why it is important to have a concept of what are measureable sets and what are non-measurable sets.

So, the non-measurable sets will fail to have this finite additivity property and the measurable sets will be the ones which are well behaved in terms of additivities, not only finite additivity, but even countable additivity as we will see in our next lectures. (Refer Slide Time: 10:14) for any $\epsilon > 0$ $\quad |[0,\epsilon]| = |[0,2]|$

Geometric intuition for subsets of $\mathbb{R}^n$:

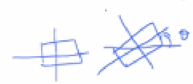
for any $E \subseteq \mathbb{R}^n \rightsquigarrow$ assign a numerical value

which corresponds to geometric intuitions:

(i) Finite additivity: If $E, F \subseteq \mathbb{R}^n$ disjoint

then $m(E \cup F) = m(E) + m(F)$

(ii) Invariance under rotations, translations and reflections.



Geometric intuitions for subsets of $\mathbb{R}^n$: For any $E \subseteq \mathbb{R}^n$, assign a (finite) numerical value, which corresponds to our geometrical intuitions: So, what are the geometric intuitions that we have?
(i)Finite additivity: If $E, F \subseteq \mathbb{R}^n$ disjoint, the measure which I denote here by m (which is the numerical assignment), then $m(E \cup F) = m(E) + m(F)$.
(ii) Invariance under rotation, translation and reflection: Which means that if you have some subset $E \subseteq \mathbb{R}^n$, and if you rotate it by some angle θ, then the area should not change. Area is invariant under rotations. Similarly for translations and reflections. (Refer Slide Time: 13: 01)

Q: Whether one can assign a numerical value $m(E)$ to any subset $E \subseteq \mathbb{R}^n$ such that (i) + (ii) are satisfied.

A: No.

Thm: (Banach-Tarski Paradox) Given a solid ball $B \subseteq \mathbb{R}^3$, there exists a decomposition of B into finitely many pieces such that they can be reassembled to produce two disjoint copies of B.

$\rightsquigarrow$ violation of finite-additivity

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So, the question is: Whether one can assign a numerical value $m(E)$ to any subset $E \subseteq \mathbb{R}^n$ such that both these conditions (i) and (ii) of finite additivity and invariance under rotations and translations and reflections are satisfied. The answer is in the negative. This is due to a theorem of Banach and Tarski.

Theorem 0.3 (Banach-Tarski Paradox). *Given a solid ball $B \subseteq \mathbb{R}^3$, there exists a decomposition of B into finitely many pieces such that they can reassembled to produce two disjoint copies of the ball B.*

This will lead to the violation of our finite additivity property. (Refer Slide Time: 15: 22)

Violation of Finite additivity due to Banach-Tarski

$$E \subseteq \mathbb{R}^n \rightsquigarrow \mu(E) \in [0, +\infty)$$

$$B = \bigsqcup_{i=1}^{n} E_i \quad \text{— (1)}$$

$$B_1 \sqcup B_2 = \bigsqcup_{i=1}^{n} F_i \text{ — (2)}, \quad B_i \cong B$$

$$\mu(F_i) = \mu(E_i), \quad i = 1, 2, \ldots, n$$

From (1): $\mu(B) = \sum_{i=1}^{n} \mu(E_i)$

$$2 \cdot \mu(B) = \mu(B_1 \sqcup B_2) = \sum_{i=1}^{n} \mu(F_i) = \sum_{i=1}^{n} \mu(E_i) = \mu(B)$$

Now, let us see how the Banach-Tarski paradox leads to the violation of finite additivity that we described before. So first of all, let for any $E \subseteq R^n$, we denote by $\mu(E)$, the measure of E. This is a value in $[0, \infty]$, which is the numerical assignment of the measure of E. Now, the Banach-Tarski paradox says that the ball B can be written as $\overset{n}{\underset{i=1}{\sqcup}} E_i$ (the disjoint union of a finite number of sets E_i, $i = 1$ to n), i.e.,

$$B = \bigsqcup_{i=1}^{n} E_i \tag{1}$$

Also, there exists $B_1 \sqcup B_2$, which can also be written as a finite union $\overset{n}{\underset{i=1}{\sqcup}} F_i$ i.e.,

$$B_1 \sqcup B_2 = \bigsqcup_{i=1}^{n} F_i, \tag{2}$$

and each B_i is a copy of the unit ball B. If we try to apply finite additivity, we quickly get a contradiction.

Here by reassembling E_i to form $B_1 \sqcup B_2$, we mean that these F_i's have the same measure as the E_i's, for $i = 1, 2, \cdots, n$. So, F_i's can be formed for example by a sequence of rotations, translations and reflections of the E_i. Since we have assumed that the measure remains invariant, $\mu(F_i)$ should be equal to $\mu(E_i)$, i.e. $\mu(F_i) = \mu(E_i)$, for $i = 1, \cdots, n$.

From Eq. (1), we have $\mu(B) = \sum_{i=1}^{n} \mu(E_i)$. On the other hand,

$$
\begin{aligned}
2\mu(B) =& \mu(B_1 \cup B_2) \\
=& \sum_{i=1}^{n} \mu(E_i) \quad \text{(by Eq. (2))} \\
=& \sum_{i=1}^{n} \mu(F_i) \quad (\text{since } \mu(F_i) = \mu(E_i)) \\
=& \mu(B).
\end{aligned}
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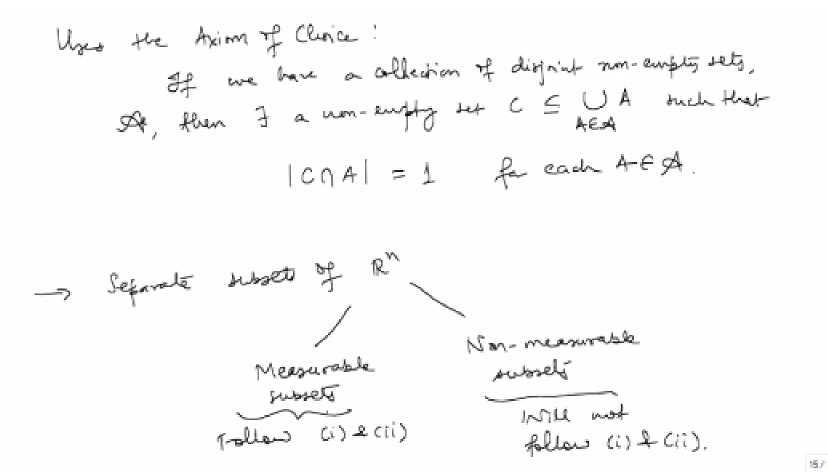
Therefore, either $\mu(B)$ be 0 or it should be infinite. Therefore, we get a violation for the finite additivity property as a consequence of the Banach-Tarski paradox. (Refer Slide Time: 19:00)



Now the Banach-Tarski paradox uses the axiom of choice. So as a consequence of the axiom of choice, we have to separate subsets of $\mathbb{R}^n$ into two categories, one as measurable subsets and another is non-measurable subsets. The measurable subsets will follow our geometric intuitions (i) and (ii), i.e., both finite additivity as well as invariance under translations, rotations and reflections, but non-measurable subsets will not follow both those conditions (i) and (ii).

The whole idea of measure theory is to study these measurable sets, which follow certain geometric intuitions. We try to build up a theory which is consistent with our usual axioms of set theory as well as the axiom of choice. We will see in the next lecture how we can start to assign numerical values to subsets of $\mathbb{R}^n$, which follow both our finite additivity rule as well as invariants under rotations, translations and reflections.