Measure Theory

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Lecture-29

Properties of the Lebesgue measure: Inner Regularity, Upward and Downward Monotone Convergence Theorem, and Dominated Convergence Theorem for Sets - Part 2

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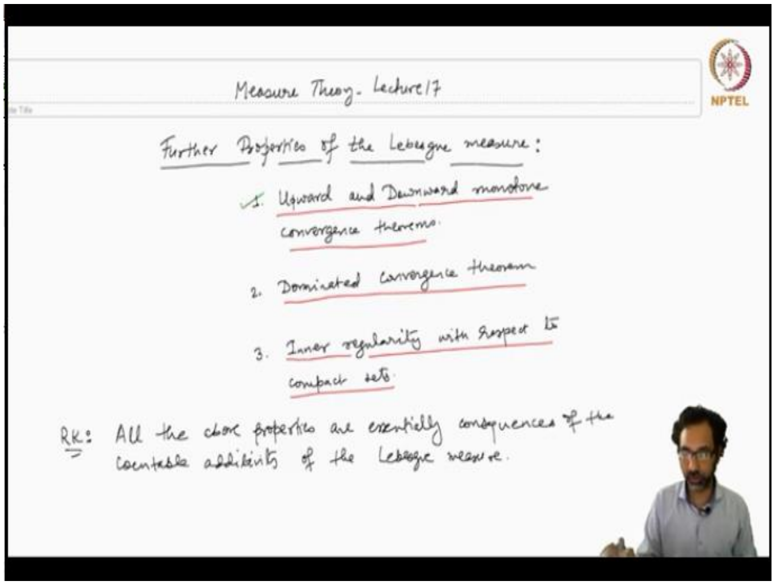

So, this proves the first property which is the upward and downward monotone convergence theorem. Now before we come to the dominated convergence theorem, I will prove the third property which is inner regularity with respect to compact sets.

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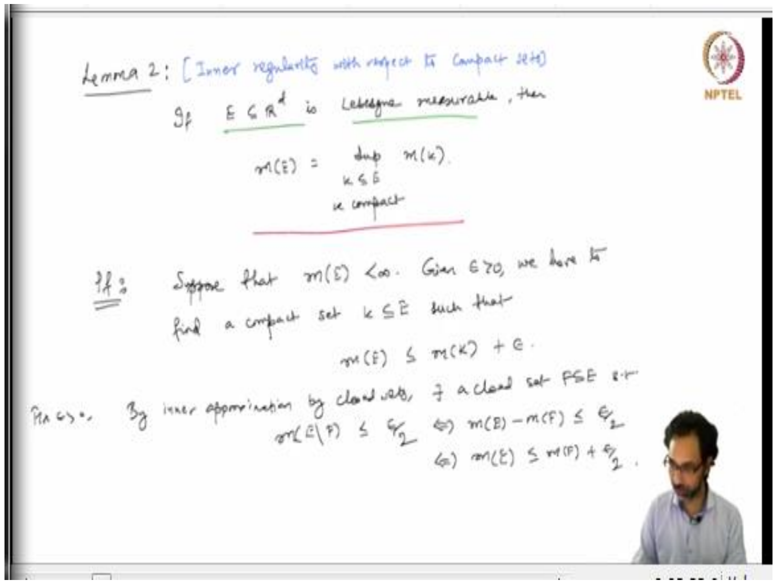

So, here is the statement for the inner regularity property with respect to compact sets. So, this says that if E is a Lebesgue measureable subset of R d, then we have this formula that m of E the measure of E can be written as the supremum of all compact subsets of E, supremum is taking over all compact subsets of E of the measures m k. So, let us prove this, so let me first suppose that the measure of E is finite and we will produce a compact set such that the supremum property is validated.

So, given epsilon greater than 0, we have to find a compact set k which is a subset of E such that m E is less than or equal to m k + epsilon. So, this will show that m E is the supremum of all these m k's. So, to do this we use the inner approximation by closed property, so by inner approximation by closed sets there exists a closed set F inside E such that the measure of E - F is less than or equal to epsilon by 2.

So, here I am fixing epsilon greater than 0 and then we use the one of these equivalent properties for Lebesgue measurability to find a close subset F of E. Such that measure of E - F is less than or equal to epsilon by 2 but this is equivalent to saying that measure of E minus measure of F is less than or equal to epsilon by 2 using finite additivity property. And because all these are finite, we can write this and so m E is less than or equal to m F + epsilon by 2.

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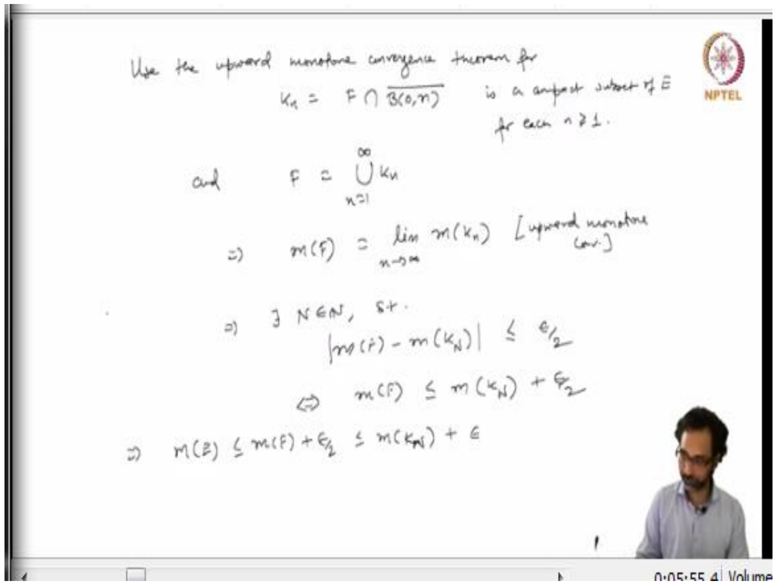

Now I am going to use the upward monotone convergence theorem for the sets k n which is F intersection the closed wall with center 0 and radius n. So, now this k n is a compact subset of E for each n greater than or equal to 1. And we also have that F is equal to the union of these k n's for n equal to 1 to infinity. So, this implies that the measure of F itself is equal to the limit as n goes to infinity of the measure of this compact sets k n.

And so this is by the upward monotone convergence and this implies that there exists. So, by the definition of the limit there exists a capital N belonging to N such that measure of F the modulus of measure of F minus the measure have k n is less than or equal to epsilon by 2. But k capital N is a subset of F, so we can get rid of the modulus sign and so this means that m F is less than or equal to m of k n + epsilon by 2.

But remember that m E was less than or equal to m F + epsilon by 2 but now we have bounded m F by m k n + epsilon by 2 + epsilon by 2 and these two epsilon by twos make an epsilon. So, we have found a compact subset of E k capital N such that our supremum condition holds, so this implies that.

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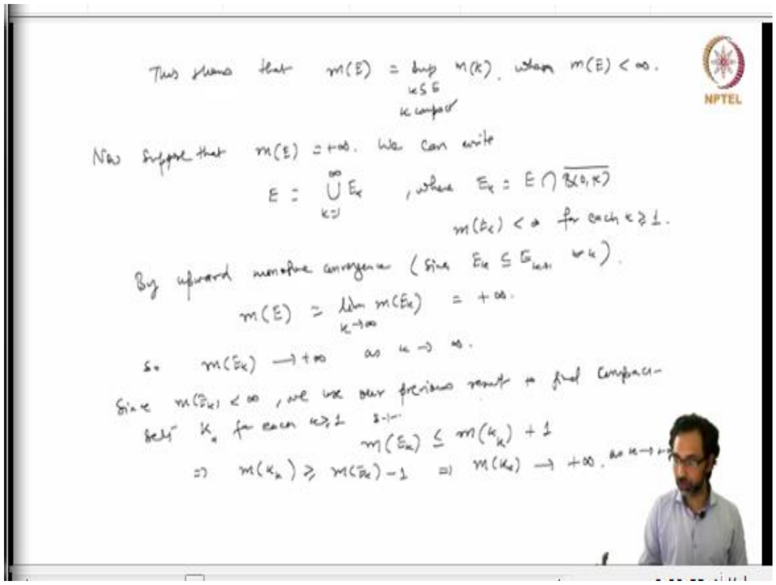

So, this shows that m E is the supremum of compact subsets taken over compact subsets of E of the measures m k when you have m E is finite. So, now suppose that m E is infinite, so then we can again write it, we can write E as the union of sets E k, k equal to 1 to infinity where the E k is the intersection of E with the closed wall of radius k with center 0. And so, now m E k is finite for each k greater than or equal to 1.

And by the upward monotone convergence again upward monotone convergence because we have a nested sequence E k is a subset of E k + 1 for all k. So, by upward monotone convergence we have that the measure of E is the limit of these measures E k as k goes to infinity and we know that this is plus infinity. So, m E k goes to plus infinity as k goes to infinity.

Now since each m E k is finite, since m E k is finite we use or previous result as we have proved here. That when m is finite then it is a supremum of the measures of compact subsets, previous result to find compact sets K k for each k greater than or equal to 1 such that m E k is less than or equal to m K k + 1 say. So, because of the supremum property for inner regularity property for measures of Lebesgue measurable subsets of finite measure we have this inequality.

This means that m K k is greater than or equal to m E k - 1 but this implies that m K k itself goes to plus infinity as k goes to plus infinity. So, we have found a sequence of compact sets subsets of E such that it is measure goes to plus infinity.

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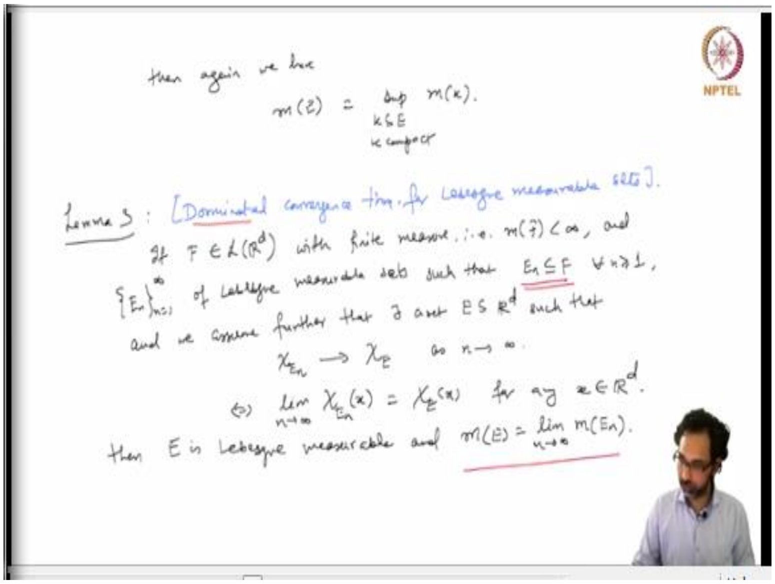

So, then again we have that the measure of E is the limit or the supremum of compact subsets of E m k because the right hand side is again now infinity. So, these proofs are inner regularity property. Now the last lemma is the so called dominated convergence theorem Lebesgue measurable sets which is the following. So, this states that if F is a Lebesgue measurable subset of R d with finite measure which is that m F is finite.

And we take a collection E n, n equal to 1 to infinity of Lebesgue measurable subsets such that each of these E n is a subset of F for all n greater than or equal to 1. And we assume further that there exists a set E subset of R d. Such that the indicator functions of these E n's converges point wise to the indicator function of E as n goes to infinity. This means that the limit as n goes to infinity chi E n x is equal to chi E x for any x in R d.

So, if we assume further that the indicator functions converge point wise to E. Then E is Lebesgue measurable, this is the first part of the claim and the measure of E is the limit as n tends to infinity of m E n. So, this is called the dominated convergence because of this domination by this set of finite measure F and we will see that this is a special case for the dominated convergence theorem for the Lebesgue integral which we will see later.

So, the domination part comes from this E n's being a subset of a fixed Lebesgue measurable set of finite measure. And the convergence holds that when you take the limit of the measures of m of E n's, it converges to the set E for which there is a point wise convergence of the indicator functions of E n.

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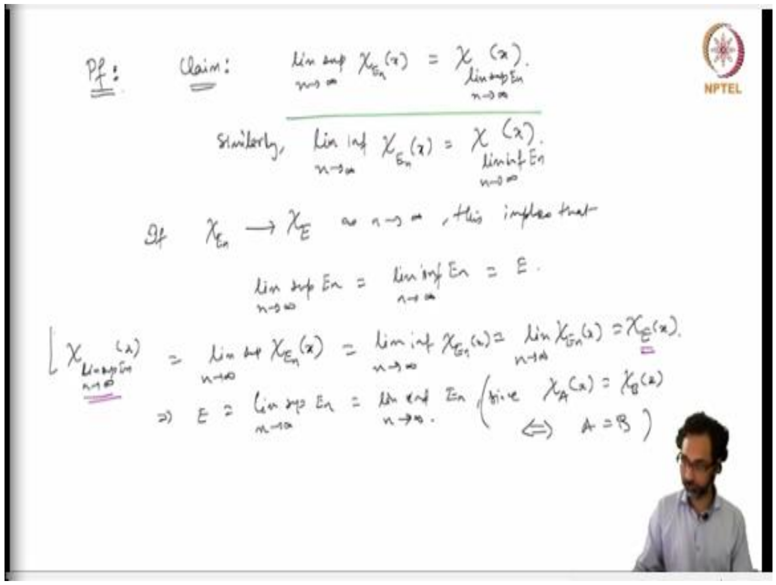

So, let us see how we prove this, so first I claim that lim sup as n tends to infinity of these functions chi E n x is nothing but chi of the lim sup of E n as n tends to infinity. So, the lim sup of the indicator functions is the indicator function for the lim sup of E n. Similarly lim inf as n tends to infinity chi E n x is equal to the indicator function of the lim inf as n goes to infinity at x. So, I will just prove the first one and the second one is quite similar. So, once we have this claim.

So, if chi E n converges to chi E as n goes to infinity, this implies that lim sup of E n equals lim inf as sets of E n as n goes to infinity is equal to E itself. So, this is because point wise convergence means that lim sup of this function chi E n x is equal to lim inf chi E n x is equal to the limit of chi E n x and this is chi E x. And because this is both the lim sup and the lim inf lim sup E n.

So, we have an equality of indicator functions for 2 sets, this implies that E is equal to lim sup E n equals lim inf E n. Because since chi A x equal to chi B x is the same as saying that A equal to B, so I am using this fact here because we have 2 sets E and lim sup of E n's and the indicator functions are the same. Therefore the E the sets themselves will be the same and similarly one can do this for the lim inf, so we have these 3 equalities that follow from this claim. So, let me prove the first one for the lim sup and let us see how this has proved.

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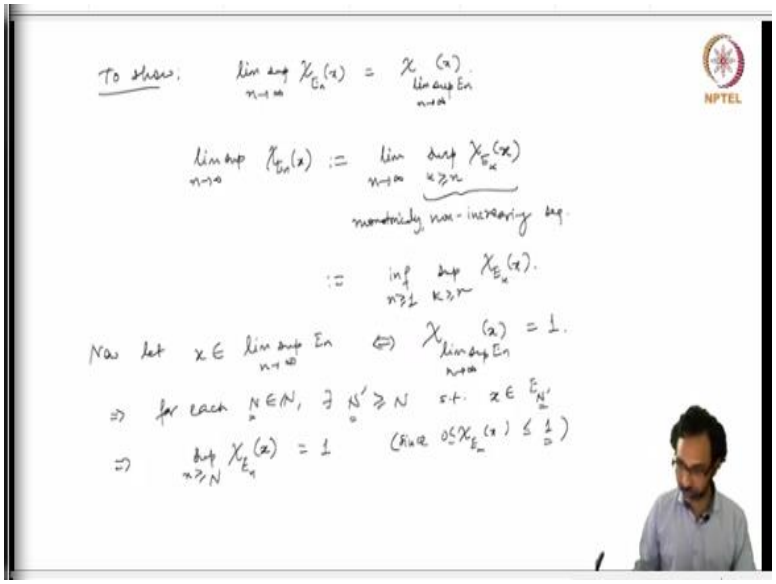

So, first let me recall the definition of the lim sup of the function chi E lim sup of chi E n x. This is by definition the limit as n tends to infinity, the supremum of k greater than or equal to n chi E k x. And this sequence this is a non increasing sequence, so monotonically non increasing sequence of numbers. So, this is also equal to the infimum of all n greater than or equal to 1 of the supremum k greater than or equal to n chi E k x.

So, this can both be taken as the definition of the lim sup and so now we have to show that. This lim sup is equal to the indicator function for the lim sup of the sets E n. So, now let x belong to the lim sup of E n, n tends to infinity. So, this is the same as saying that the indicator function has the value 1. So, because it belongs to the lim sup of all E n's this implies that there exists, so for each N in N there exists N prime greater than or equal to N such that x belongs to E N prime.

So, for each capital N we will find a bigger N prime such that x belongs to E N prime. So, this follows directly from the definition of the lim sup. Remember that x belongs to infinitely many elements of the lim sup. So, if you take any finite N then there is always exists N prime greater than that and such that x belongs to E N prime. So, this implies that the supremum of n greater than or equal to N prime of this chi e n x or rather k N greater than or equal to N chi E n x, this is going to be 1.

Because the indicator function of any set is bounded above by 1 and bounded below by 0. And so if it belongs to E N prime for N prime greater than or equal to N, so it will assume its maximum value and so the supremum will be that maximum value 1, so this is true for any capital N.

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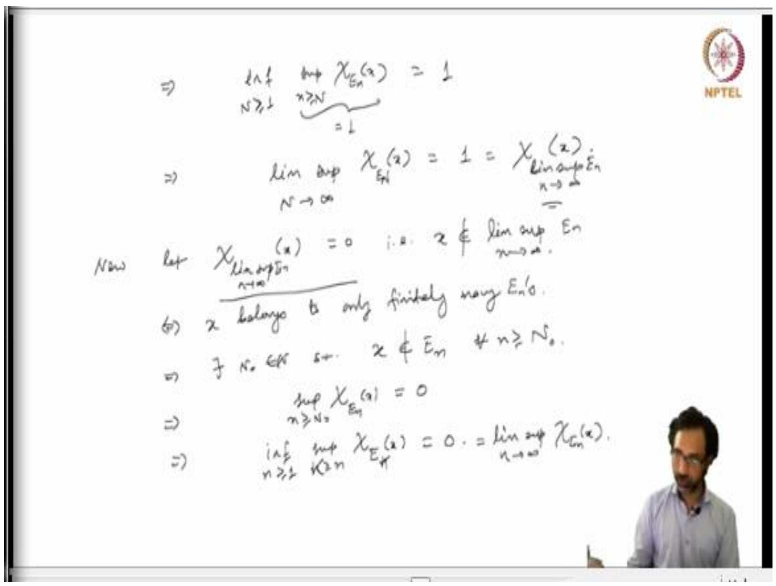

So, therefore this implies that if you take the infimum over all capital N greater than or equal to 1 of the supremum values N greater than or equal to N capital N chi E n x. So, each one this is equal to 1, so therefore the infimum is also equal to 1 but this is nothing but the lim sup as capital N goes to infinity of this numbers chi E N capital N x, so this is equal to 1. So, in this case, we have proved that this lim sup is equal to the lim sup of E n of x.

So, when the right hand side is 1, the left hand side is also 1. Now, I am going to prove that if the right hand side is 0, then the left hand side is also 0. So, because the indicator functions takes only these 2 values our analysis is easier somewhat that we only have to check for these 2 values 0 and 1. So, now let that the indicator function of the lim sup E n this is 0, so this means that x does not belong to the lim sup of E n’s.

So, this is equivalent to saying that x belongs to only finitely many E n's because the lim sup is by definition, the set for which x belongs to infinitely many such E n's. So, if it does not belong to the lim sup then x belongs to only finitely many E n's and this implies that there exists N 0 belonging to the natural numbers. Such that x does not belong to E n for all n greater than or equal to N 0.

So, after a fixed finite value N 0 x does not belong to any other E n's. So, this implies that the supremum of n greater than or equal to N 0 of chi E n x is equal to 0. Because it does not belong to any of the E n's, so it will take the constant value 0 after N 0. Which means that because these values for the indicator function is bounded below by 0. This also means that the infimum over all n of the supremum of overall k greater than or equal to n chi E k x, this is also 0.

Because after N 0 it takes the least value possible. So, when you take the infimum of all N then it must be 0 and this is nothing but again the lim sup, lim sup n goes to infinity chi E n x. So, we have shown that when this is 0 when the indicator function for the lim sup of the sets is 0, then the lim sup of the indicator functions is also 0.

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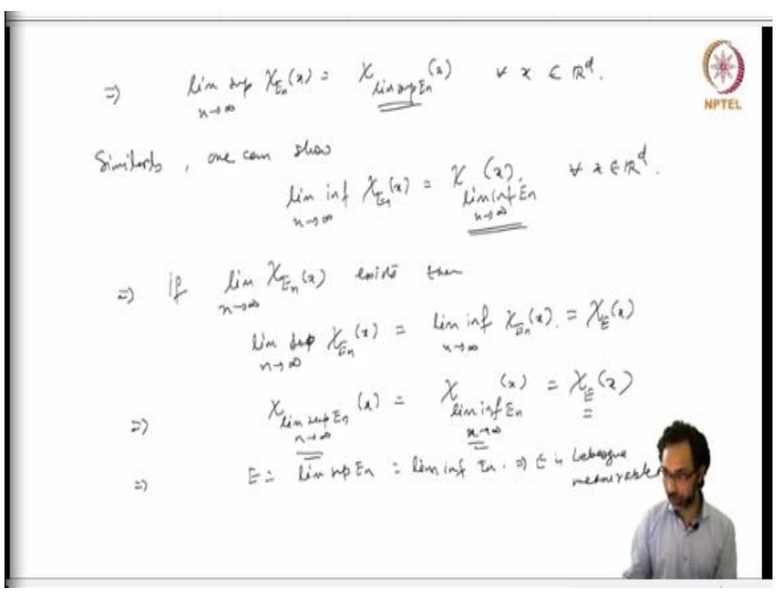

So, we have shown that lim sup this implies that lim sup chi E n x is equal to chi lim sup E n x, so for all x in R d. So, similarly one can show that the lim inf n goes to infinity chi E and x is equal to chi lim inf E n, n goes to infinity of x for all x belongs to R d. So, this you can view this as the justification of calling this set lim inf or even lim sup we defined it using unions and intersections. But this is somewhat a justification for calling them lim sup and lim inf which is not evident from the basic definition of lim sup and lim inf.

So, in this way we have identified the indicator functions. So, this implies that if limit n tends to infinity chi E n x exists, then lim sup of n tends to infinity chi E n x is equal to the lim inf n tends to infinity chi E n x. Both of these are bounded and the limit exists precisely when these two are equal. So, therefore this implies that chi of the lim sup of E n x is equal to chi of lim inf of E n as n tends to infinity.

And by the hypothesis this is also equal to chi E of x, so we have three indicator functions for three different sets lim sup E n lim inf E n and E. But indicator functions are the same means the sets themselves are the same. So, E is equal to lim sup E n is equal to lim inf E n and this is why E is Lebesgue measurable. Because the lim inf and lim sup of measurable functions are Lebesgue measurable, so E is itself Lebesgue miserable.

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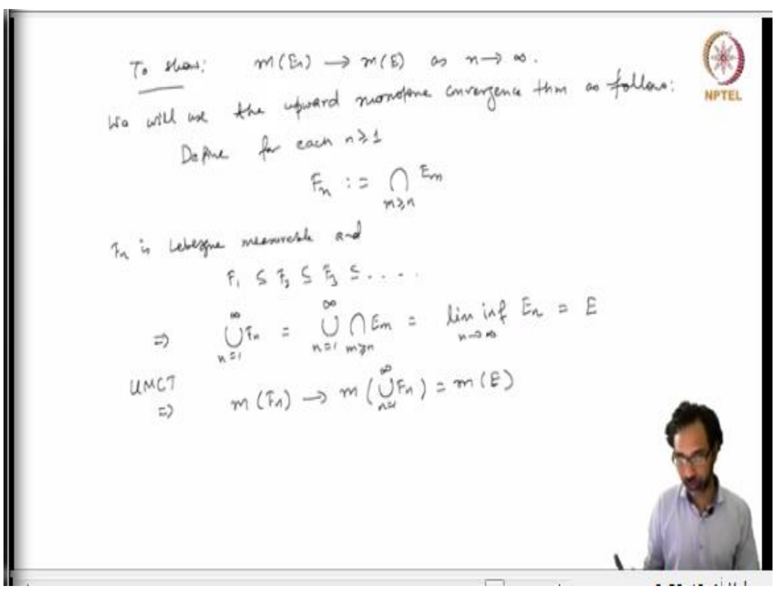

So, now we have to show that m E n converges to m E as n tends to infinity. So, we are going to use the upward monotone convergence theorem. So, we will use the upward monotone convergence theorem as follows. So, for this we define for each n greater than or equal to 1 F n to be the intersection of all the E n's. So, this is a Lebesgue measurable set F n is Lebesgue measurable and we have F 1 is a subset of F 2 is a subset of F 3 and so on. So, it is a non decreasing **c** sequence of nested Lebesgue measurable sets.

So, therefore if you take the union of F n's, n equal to 1 to infinity, this is nothing but the union of the intersections E m. And this is exactly the lim inf as n tends to infinity of E n's and this we know that this is E. Therefore by the upward monotone convergence theorem we get that m F n converges to m union F n, n equal to 1 to infinity and this is m E.

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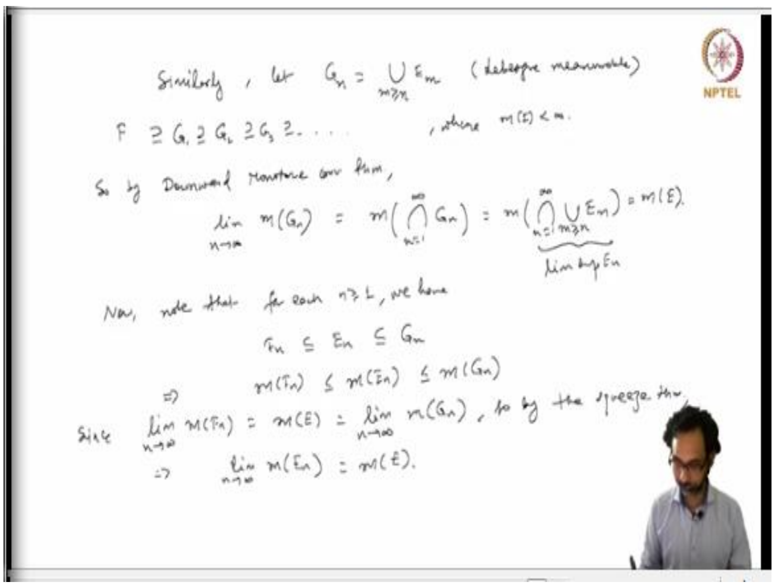

Similarly for the sets which we will define as G n these are the unions n greater than or equal to n E m. Then these G n's these are Lebesgue measurable and we have G 1 which is superset of G 2 which is superset of G 3 and so on. And each of these G i's is contained in F where F is has finite measure. So, one can apply the downward monotone convergence theorem, so by the downward monotone convergence theorem we have that the limit as n tends to infinity the measure of G n.

So, this limit is nothing but the measure of the intersections of G n n equal to 1 to infinity but this is nothing but n equal to 1 to infinity union m greater than or equal to n E n. And this is the lim sup of E n and so this is again m of E. So, now notice that we have for each n we have F n is a subset of E n is a subset of G n. So, this implies that measure of F n is less than or equal to measure of E n is less than or equal to measure of G n.

So, since the limit on both sides of inequality limit F n measure of F n is equal to measure of E which is also equal to the limit of measure of G n. So, by the squeeze theorem for limits of sequences we get, this implies that the limit n tends to infinity measure of E n is also equal to the measure of E.