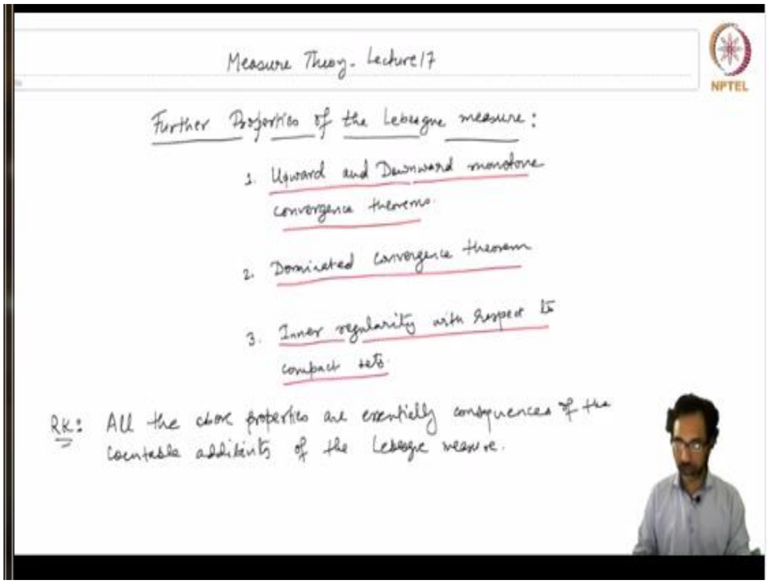
Measure Theory
Prof. Indrava Roy
Department of Mathematics
Institute of Mathematical Science

Lecture-28

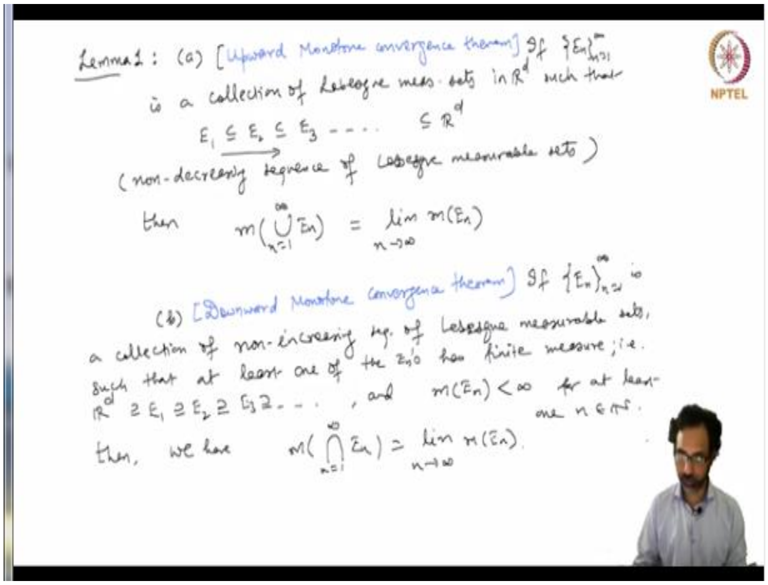
Properties of the Lebesgue Measure:Inner Regularity, Upward and Downward Monotone Convergence Theorem, and Dominated Convergence Theorem for Sets-Part 1

(Refer Slide Time: 00:14)



So, we continue our study of the Lebesgue measure and in this lecture we will look at three main properties of the Lebesgue measurable sets. The first one is called upward and downward monotone convergence theorems and the second one is the dominated convergence theorem and the third one is called inner regularity with respect to compact sets. So, all of these, so let me put this as a remark that all these properties all the above properties are essentially consequences of the countable additivity property of the Lebesgue measure, so let me state these properties as lemmas.

(Refer Slide Time: 01:26)



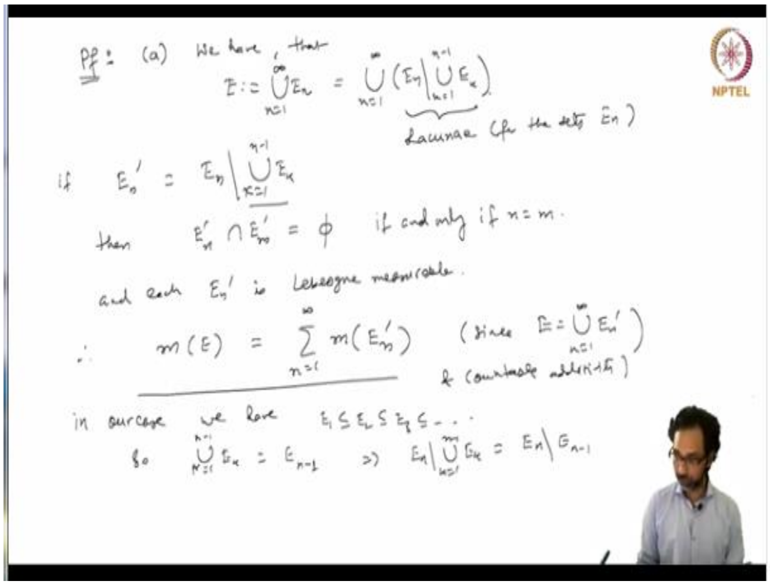
So, the first one, so let me call it lemma 1 a, this is the upward monotone convergence theorem and this states that if E n, n equal to 1 to infinity is a collection of Lebesgue measurable sets in R d such that we have a nested sequence of non decreasing sets E 1 is a subset of E 2 is a subset of E 3 and so on, then all of these are subset of R d. So, this is a non decreasing sequence of Lebesgue measurable sets then the measure of the union n equal to 1 to infinity E n which is again Lebesgue measurable.

So, the measure of the countable union of all these E n's is equal to the limit as n goes to infinity of the sequence of measures m E n. So, this is the upward monotone convergence theorem, this is upward monotone means that you are going higher and higher in the nested sequence of sets. And monotonicity property of the Lebesgue measure gives the name monotone convergence theorem, so this is the upward monotone convergence theorem.

The second one is the downward monotone convergence theorem, so this states that if again E n is a collection of non increasing. So, this is the reverse way sequence of Lebesgue measurable sets such that at least one of the E n’s has finite measure. So, in symbols we have E 1 is a superset of E 2 is a superset of E 3 and so on. And all of these are subsets of R d and m E n is finite for at least one n in the natural numbers.

So, then we have that the measure of the intersection now of all these E n's, n equal to 1 to infinity is equal to the limit n tends to infinity of the sequence m E n. So, notice that, in the first in the part A there is no assumption for finiteness of measure of any of the E n’s. But in the second one, we do need that m E n is finite for at least one of the E n’s, so let us see the proof of this lemma.

(Refer Slide Time: 05:44)



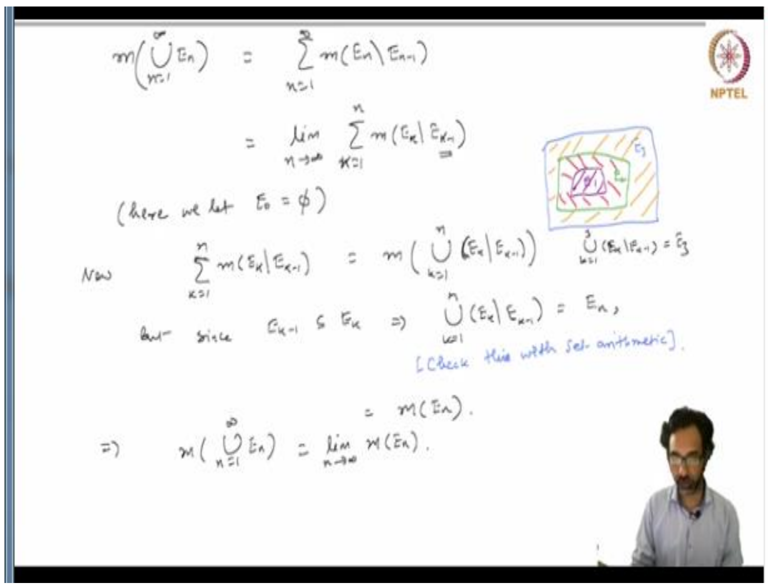
So, for part a we have that the union of all these this can be written as, so let me denote this by E. And this can be written as a disjoint union of sets n equal to 1 to infinity, E n - union k equal to 1 to n - 1 E k. So, these are called Lacunae for the sets E n and this makes each of this Lacunae disjoint from any other Lacunae. So, this also gives you an increasing family of sets. So, let me write if E n prime is equal to this E n - this Lacunae k equal to 1 to n - 1 E k.

Then E prime m intersection E prime m this is equal to the empty set is an only if n equal to m. So, this is the collection of E n primes is pair wise disjoint say disjoint collection of sets and each of these each E n prime is Lebesgue measurable. Because the finite union of the E k’s is measurable and the complement is measurable and then again there is an intersection with a Lebesgue measurable set E n this is also measurable.

So, for each E n prime is Lebesgue measurable. So, therefore we have the measure of E is equal to the sum n equal to 1 to infinity m E n prime. Because since E is a disjoint union of all these

Lebesgue measurable sets. We have from countable additivity property we have that the measure of E is just the sum. So, this is for any collection of countable collection of Lebesgue measurable sets E n. But in our case, we have a non decreasing sequence E 1 is a subset of E 2 is a subset of E 3 and so on. So, the union of k equal to 1 to n - 1 E k is nothing but E n - 1. So, in our case this Lacunae E n - k equal to 1 to n - 1 E k this is nothing but - E n - 1. So, once we have this formula we can rewrite it.

(Refer Slide Time: 09:39)



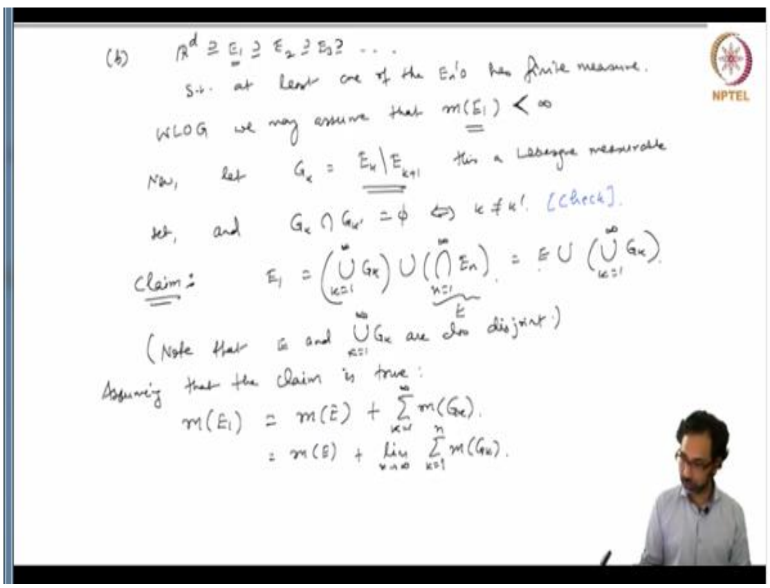
So, now the for our case the union of all these E n's when you have a non decreasing sequence this is equal to the sum of the measures m E n - E n - 1, n equal to 1 to infinity. And this is by definition the limit of all these partial sums 1 to k equal to 1 to n m E k - E k - 1 this is by definition of the infinite sum. Here we let E 0 to be the empty set because if k equal to 1 we will get E 0 but our sequence of sets starts from E 1.

But we can without loss of generality right E 0 to be the empty set and so now this limit formula makes sense. So, now we look at these partial sums k equal to 1 to n m E k - E k - 1 because each of these for each k this is a collection of disjoint sets the Lacunae are disjoint. So, this is nothing but the union k equal to 1 to n E k - E k - 1 but since E k - 1 is a subset of E k. This implies that the union k equal to 1 to n E k - E k - 1 is nothing but E n, because, so this is something like this as a Venn diagram, you have E 1, then you have E 2, then you have E 3 and so on 3.

And so for example here if you take union k equal to 1, 2, 3 E k - E k - 1. So, the first part is nothing but for k equal to 1 this is nothing but whole of E 1 then you will get this part in red E 2 - E 1 and then you will get E 3 - E 1. So, in total you will get nothing but the last term of the series which is nothing but E 3. And similarly you can check this with set that set arithmetic again that indeed the union of these Lacunae will give you E n.

So, therefore on the right hand side we get m E n and so we get our required result that thus union has measured given by the limit of m E n. Notice that we did not assume any finiteness of the measures because we are not cancelling anything from both sides of the equation. So, we can all the arguments that we give works whether each of these measures are finite or infinite.

(Refer Slide Time: 13:56)



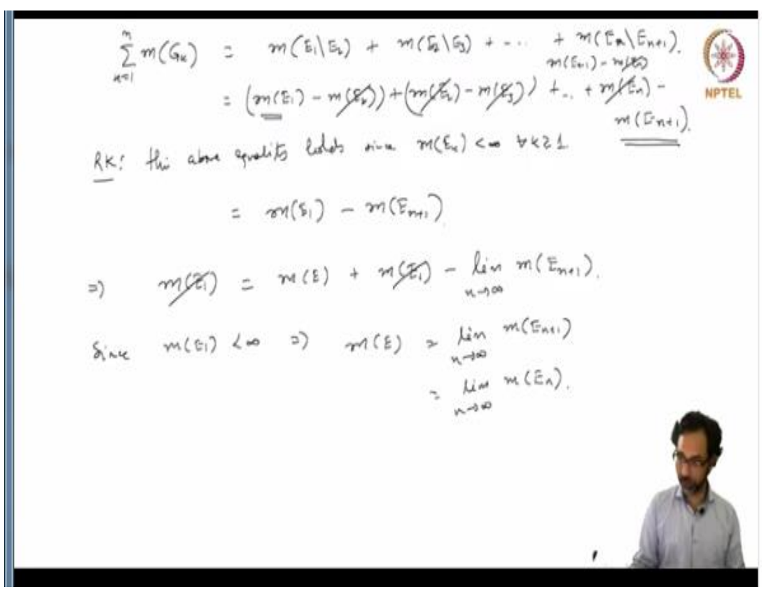
Now let us see part b, here we have a non increasing sequence of nested Lebesgue measurable sets E 3 and so on and such that at least one of the E n's has finite measure. So, know that without loss of generality, we may assume that the measure of E 1 itself is finite, otherwise there can be renumbering of the sequence and the limit remains unchanged. So, we can assume that the first one E 1 has finite measure.

So, now let again G k be the set E k - E k + 1 here because it is a decreasing or other non increasing family of sets. So, earlier we took E k - E k - 1 but here we have to take k + 1. So, this is a Lebesgue measurable set and again this family of G k's is pair wise disjoint meaning that G k and G k prime have empty intersection if and only if k is not equal to k prime. So, you can check this using again set arithmetic that indeed this is the case, so now I make a claim. So, let me write it as a claim I claim that E 1 can be written as the union k equal to 1 to infinity G k union with the intersection of n equal to 1 to infinity E n.

So, let me write this as E, so this is E union k equal to 1 to infinity G k. So, now note that E and this union k equal to 1 to infinity G k are also disjoint ok. So, E 1 can be written as a disjoint union of a countable disjoint union of sets E and G k's. And so if we assume that the claim is true then we have that the measure of E 1 is equal to measure of E plus the sum k equal to 1 to infinity measure of G k and G k was the set E k - E k + 1.

So, if you write this as measure of E plus the limit as k tends to infinity or rather n tends to infinity k equal to 1 to n m G k. So, this is by definition of the infinite sum again, so now we are going to see what these partial sums exactly are, so let us see.

(Refer Slide Time: 18:26)

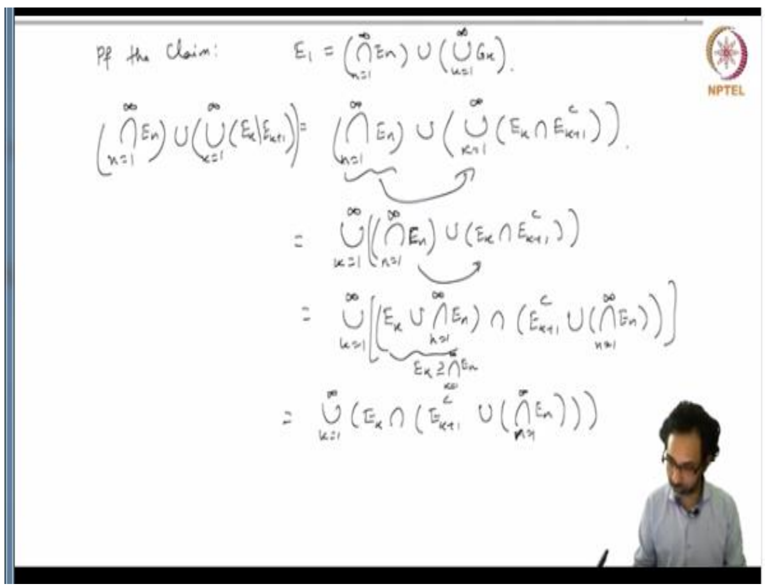


So, this sum k equal to 1 to n m G k this is nothing but the first term is m G 1 which is m E 1 - E 2 + m E 2 - E 3 and so on, so the last term is m E n - E n + 1. So, because all of these are finite, we can write this as m E 1 - m E 2 + m E 2 - m E 3 and so on and the last term is m E n - m E n + 1. So, I will put this as the remark that this above equality holds since m E k is finite for all k greater than equal to 1, so now notice that we have many cancellations.

So, here we have m E 2 - E 2 and + m E 2 cancel out, similarly - m E 3 will cancel out with the next term which will contain + m E 3 and so on. And m E n will also cancel out with the last term because the n - 1 term is m E n - 1 - m E n. So, this term and this term will cancel out but we will be left with m E n + 1 and m E 1, so this is nothing but m E 1 - m E n + 1. So, therefore we have that measure of E 1 is equal to measure of E + measure of E 1 - measure of the limit n goes to infinity m E n + 1.

So, since again m E 1 is finite this implies that we can cancel out m E 1 on both sides and so m E is equal to the limit n goes to infinity m E n + 1 which is the same as the limit n goes to infinity m E n, so we have done except that we have to prove the claim.

(Refer Slide Time: 21:32)



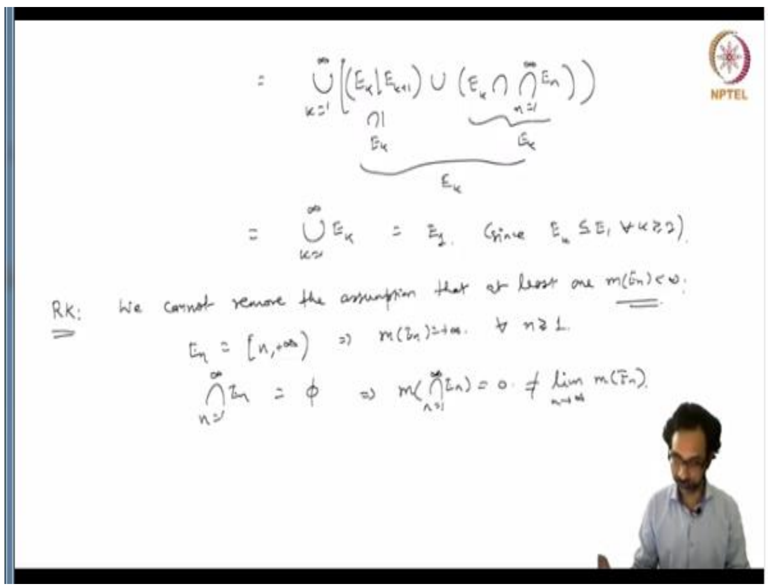
So, now, let us proof for the claim which stated that E 1 is the union of the intersection of all E n's and the union with the union of all the G k's k equal to 1 to infinity G k. So, this is nothing but so let me take this right hand side n equal to 1 to infinity union with the k equal to 1 to infinity and G k is E k - E k + 1. And so this is equal to the intersection n equal to 1 to infinity E n union with a union k equal to 1 to infinity E k intersection E k + 1 compliment.

And now we can distribute this set and take this inside the big union. So, this is k equal to 1 to infinity intersection n equal to 1 to infinity E n. And then you have a union with E k intersection

E k + 1 compliment and then again we can distribute this inside the intersection. So, you will get E k union of the intersection n equal to 1 to infinity E n. And then there is an intersection with the E k + 1 complement union with the intersection n equal to 1 to infinity E n.

Now notice that the first one is just E k because this intersection is contained inside E k for all k. And so we have k equal to 1 to infinity E k intersection with this second term which is E k + 1 union with the intersection of all E n's. Now we can again distribute this E k inside, so we will get.

(Refer Slide Time: 24:18)



So, here we have k equal to 1 to infinity E k intersection E k + 1 complement which is nothing but E k - E k + 1. And then you have a union with E k and then the intersection with the all the intersection of all E n's. But this is again E k and this is a subset of E k, so the union is again E k. So, therefore you get k equal to 1 to infinity E k but since it is a non increasing sequence, this is nothing but E 1.

Since E k is a subset of E 1 for all k greater than or equal to 2. So, therefore we have proved our claim and so this proves the downward monotone convergence theorem. One remark about the downward monotone convergence theorem is that we cannot remove the assumption that at least one m E n is finite. So, as this example shows, so you can take E n to be the set n 2 + infinity, so it is an unbounded set.

So, it is a Lebesgue measurable set and it has measure + infinity, in fact this is an elementary set. So, the measure is + infinity for all n greater than or equal to 1, so the intersection of all the E n's is empty, so the measure of the intersection is 0. So, this is not equal to the limit of the measures m E n. So, as this example shows that we cannot remove the assumption that at least one of them has finite measure.