Measure Theory
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Lecture-27
The Measure Axioms and the Borel-Cantelli Lemma

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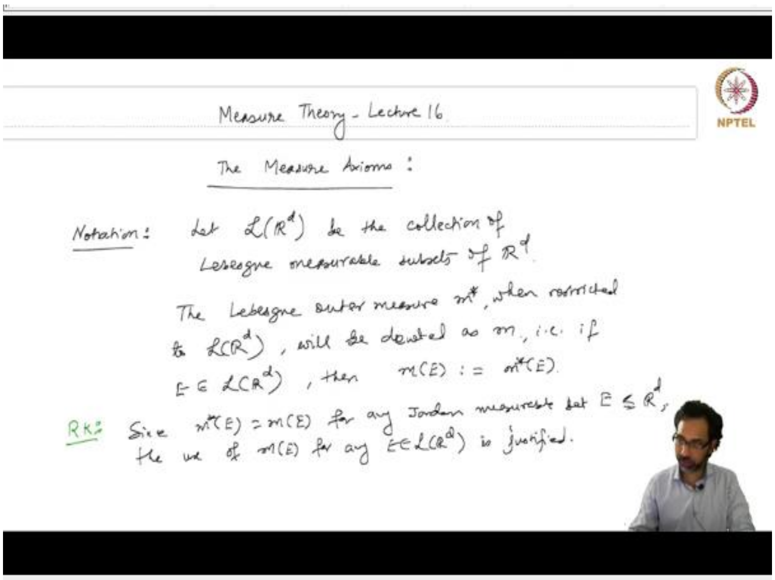

In this lecture we will state the so called measure axioms which give the property of countable additivity of Lebesgue measurable disjoint collection of Lebesgue measurable sets. So, this gives us the right framework for talking about a measure, which is not only finitely additive for disjoint measureable sets, but also countable additive meaning that if you have a countable collection of disjoint Lebesgue measurable sets.

Then the measure of the union of all those sets should be infinite sum of the measures of the individual sets in the union. So, before I state the measure axioms let me state some notation. So, I denote the calligraphic L of R d as the collection of Lebesgue measurable sets, subsets of R d and the Lebesgue outer measure m star when restricted to this collection of Lebesgue measurable subsets will be as denoted as m.

So, which means that if E is a Lebesgue measureable subset of R d, then m f of E is by definition m star of E. So, we have already used this notation m of E for the elementary measure and for the Jordan measure, but since m star of E. So, this is a remark that since m star of E is equal to m E for any Jordan measurable set. This is one of the results that we proved earlier that if E is any Jordan measurable subset of R d.

Then the Lebesgue outer measure is equal to the Jordan measure of E. So, the use of m E for any E in the collection of Lebesgue measurable subsets of R d is justified and it is in fact an extension of the notation used for Jordan measurable and elementary subsets of R d. So, what are the measure axioms.

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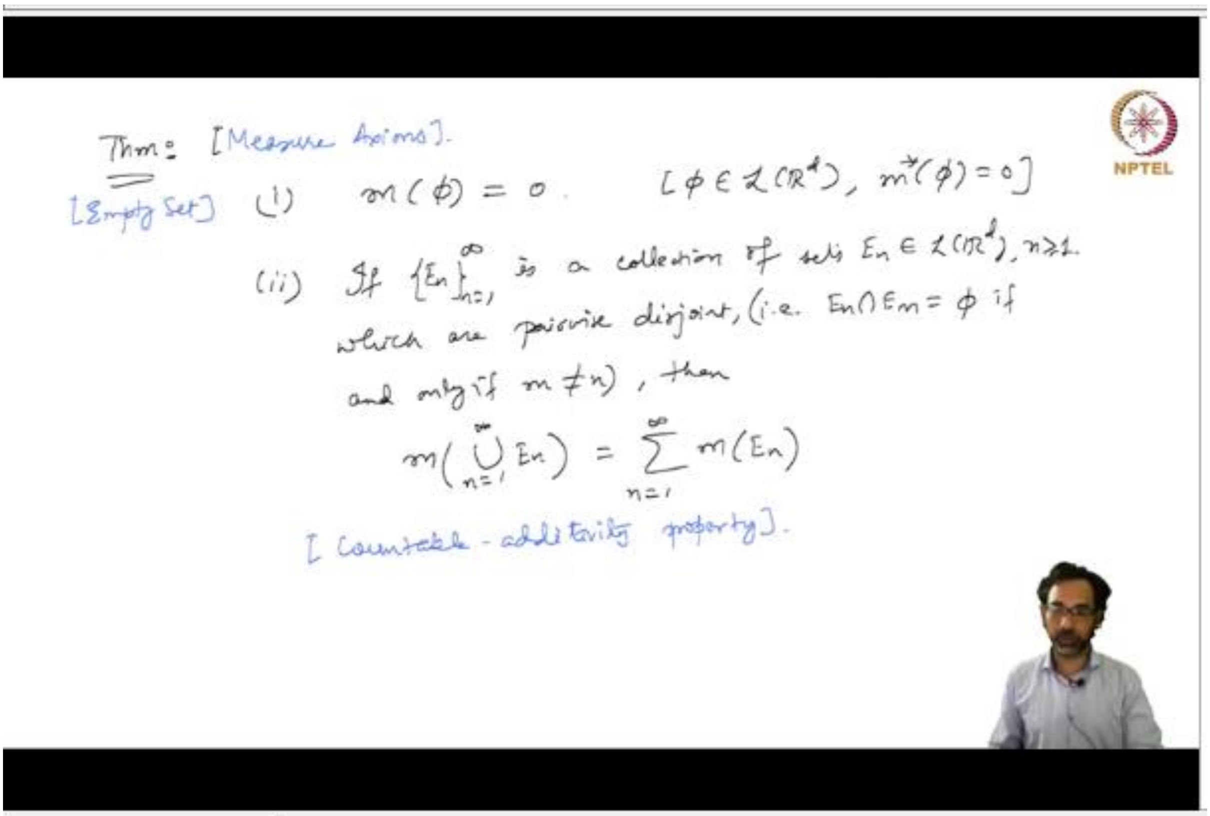

So, let me put this as a theorem the measure axioms. So, let okay. So, first that m of phi equals 0, this we have already seen that the empty set is a Lebesgue miserable. So, it belongs to this collection L of R d and m star of phi and star of phi was computed to be 0. So, since m of phi is m star a phi because phi is Lebesgue measurable m of a phi is 0s, but we have already seen this result.

But we are resetting it because it will be part of the axioms of measure when we are dealing with abstract measure spaces, then it will be part of the axiom. So, this was the first part. The second is that if E n n equal to 1 to infinity is a collection of sets E n in L R d for n greater than or equal to 1. So collections of Lebesgue measurable sets, which are pairwise disjoint, meaning that E n intersection E m is empty if and only if m is not equal to n.

So, if they are pairwise disjoint then the measure of the union n equal to 1 to infinity of E n is equal to the sum n equal to 1 to infinity m of E n. So, since everything here is Lebesgue measurable, we can use m instead of m star throughout. So, I am using both on both sides m of the sets rather than m star. So, this second property is this is known as countable additivity property and this is one of the most important properties of the Lebesgue measure.

And because the Lebesgue measure is a prototypical example of a measure, this axiom will also be part of the axioms for a measure on an abstract measure space. So, we will come to that later. And we will just state here that this is the countable additivity property of the Lebesgue measure, this is the empty set property the first one and the second one is the countable additivity property. So, first one we have already seen. So, I just proved the second one.

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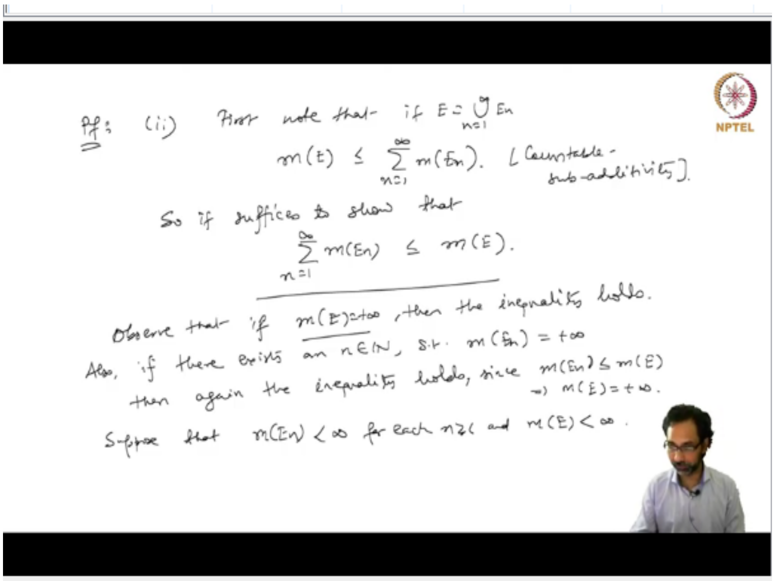

So, for the proof of the second part note that; so, first note that m of E. So, if E is the union then m of E is less than or equal to this sum of the m E n’s. So, it suffices to show that the sum n equal to 1 to infinity m E n is less than or equal to m E. So, we have to prove the reverse inequality. So, this inequality was due to countable sub additivity rather than additivity which holds for disjoint collection of Lebesgue measurable sets.

So, to prove this second inequality, we note observe that if m E is plus infinity then the inequality holds trivially also if m E n is equal to plus infinity for any n for. So, let me rewrite this as follows. So, if there exists, an n says that m E n is equal to plus infinity, then again the inequality holds. Inequality holds because m E n is less than or equal to m E. So, this implies that m E is equal to plus infinity and then we are back to the previous case. So, I assume that m E n is finite for each n and m E is also finite.

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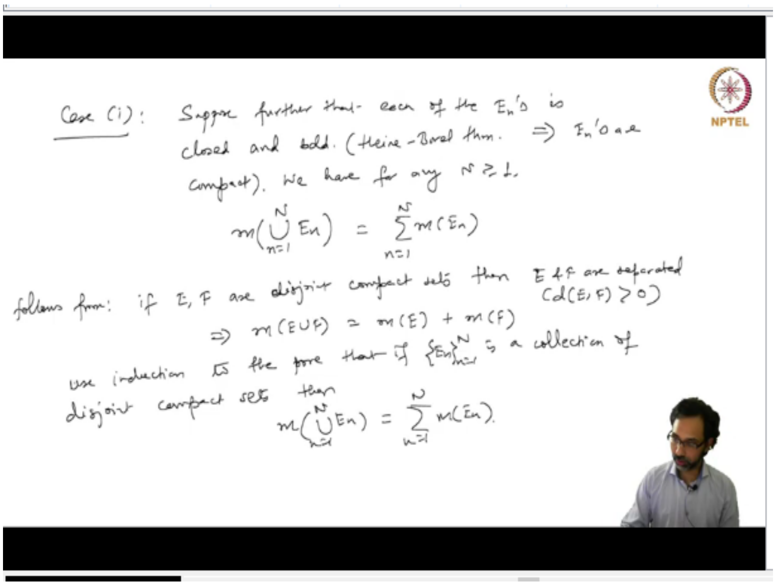

So, now I will break up this case into further sub cases. So, case 1 is that suppose further that each of the E n's is closed and bounded which by the Heine-Borel theorem means E n's are compact, Heine-Borel theorem is equivalent to saying that each of these E n’s are compact. So, in this case, we have that for any capital N greater than or equal to 1.

The measure of n equal to 1 to capital N E n is the sum n equal to 1 to n m of E n, because this is the finite additivity property for disjoint compact sets. So, we have already seen that if so, this follows from the following that if E and F are compact disjoint compact sets, then m of E union F is equal to m E plus m F, because E and F are separated**.** Remember that this means that the distance between E and F is strictly positive and it follows that the finite additivity property holds.

And an induction argument use induction to then prove that if E n and N equal to 1 to capital N is a collection of disjoint compact sets, then we have n equal to 1 to capital N E n is equal to n equal to 1 to n m of E n.

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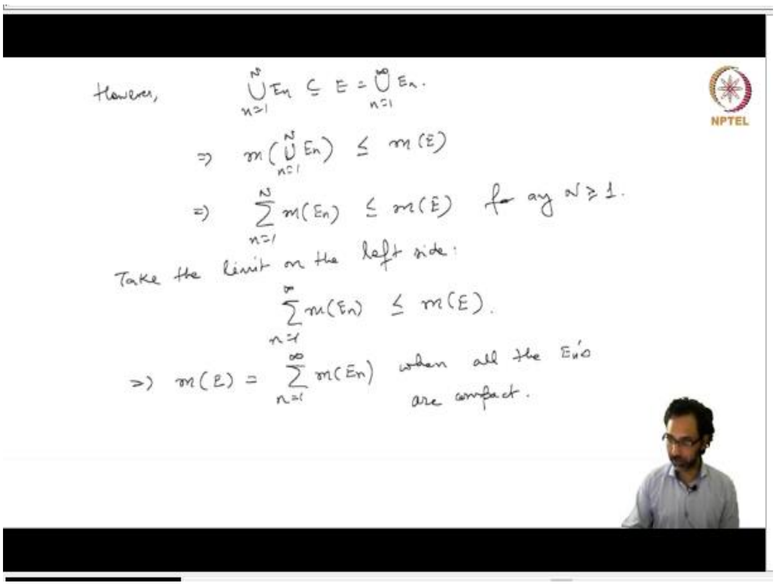

On the other hand however union n equal to 1 to n E n is a subset of E which was by definition the entire union n equal to 1 to infinity of E n which means that m E n m union n equal to 1 to infinity E n is less than or equal to m of E but the left hand side is just the sum of the m E n’s and this is for any n greater than or equal to 1. So, now take the limit on the left side to get that the sum n equal to 1 to infinity m E n is less than or equal to m of E. So, this implies that m E is equal to the sum n equal to 1to infinity m E n, when all the E n's are compact.

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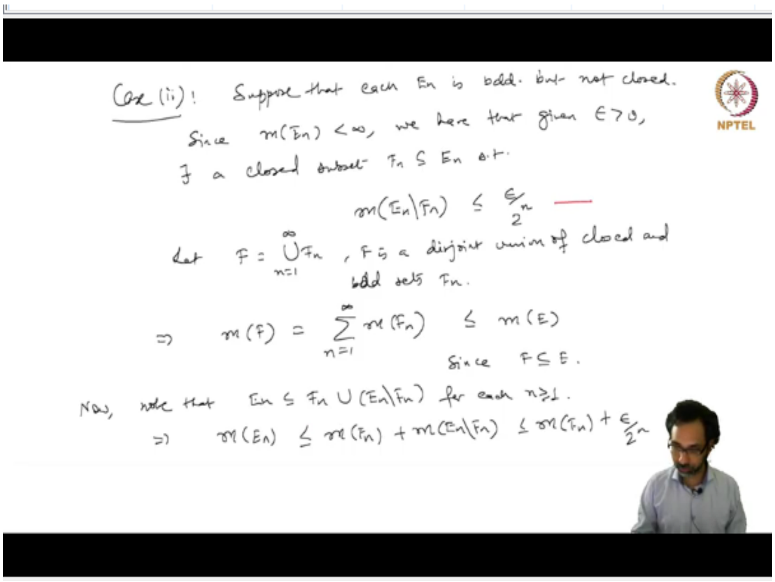

Now, the next case is case 2 is that suppose that each E n is bounded but not closed. So, first we had compactness which was closed and bounded. And now I am dropping the assumption that it is closed but I am keeping the assumption that it is nevertheless bounded So, then since m E n is finite, we have the given epsilon greater than 0, there exists a closed subset F n inside E n says that the measure of E n minus F n is less than or equal to epsilon over 2 to the power n.

So, I am again using the 2 the power n trick because I have to sum in the n. So, and let me put F as the union of all these sets F n, n equal to 1 to infinity and F is a disjoint union of closed and bounded sets and bounded sets F n . F n is close but it is also bounded because each of the E n is bounded. So, this implies that by the first case that m of F is equal to the sum of m F n and equal to 1 to infinity.

And this is clearly less than or equal to m E since F is a subset of E. So, now, note that E n is a subset of F n union E n minus F n for each n of course, and so m of E n is less than or equal to m of F n plus m of E n minus F n but the second one is less than or equal to epsilon by 2 to the power n by our assumption here.

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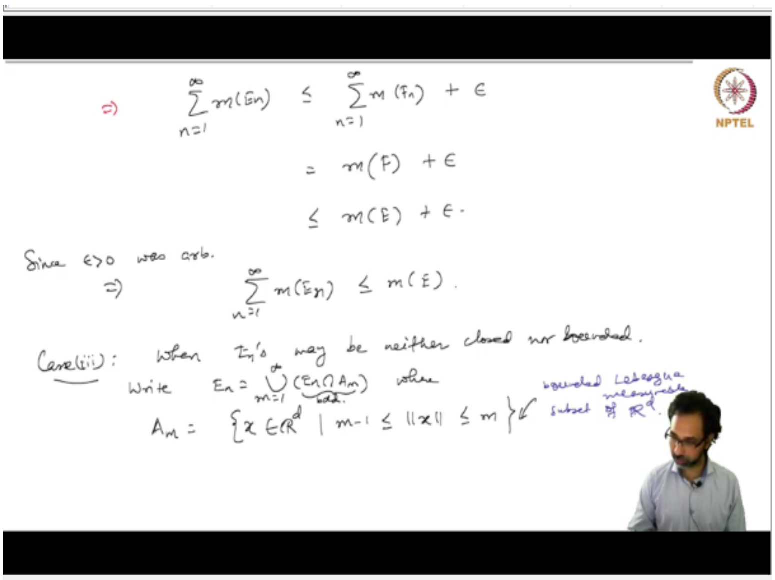

So, now, we can sum on both sides to get the sum of m E n n equal to 1 to infinity is less than or equal to the sum n equal to 1 to infinity m F n + epsilon, but this is nothing but the measure of F + epsilon and this is nothing but measure E + epsilon. So, this implies that since epsilon was arbitrary we have the desire inequality for the case when each E n's are assumed bounded, but they may not become closed.

Now, a third case is the general case, this is the case when E n's may be neither closed nor bounded. So, in this case we can make them bounded by write each E n as the countable union of sets, E n intersection A m, where A m is given by the set of points in R d says that the Euclidean norm of x is bounded between m - 1 and m. So, notice that this is a measurable bounded Lebesgue measurable subset of R d.

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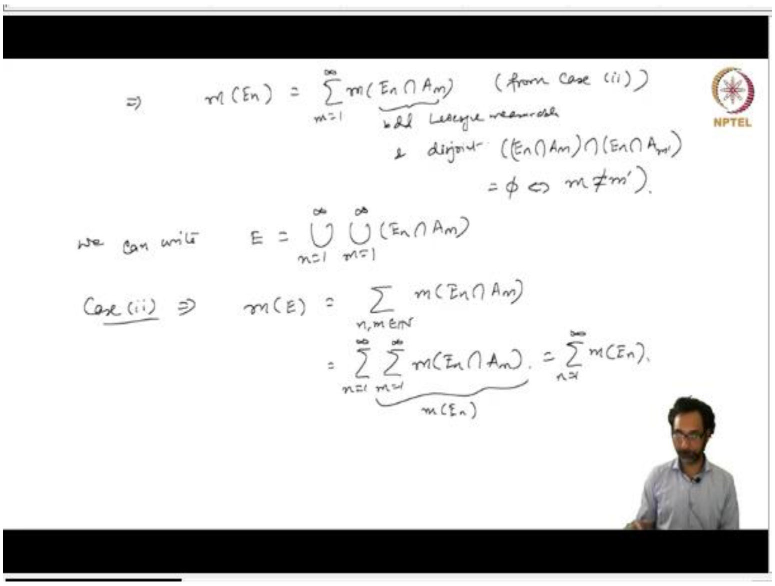

So, now this means that m E n is the sum m equal to 1 to n infinity m E n intersection A m. This is from case 2, where each of these E n intersection m, these are bounded Lebesgue measurable subsets and disjoint bounded Lebesgue measurable and disjoint, meaning that E n intersection A m intersection E n intersection A m prime is empty if and only if m is not equal to m prime.

So, we have the measure of E n is the sum of all these portions of E n in this annulus region. So, now, we can write E, this is the union n equal to 1 infinity union m equal to 1 to infinity E n intersection A m and again by the second case, we get that m of E is the double sum of the measures E n intersection A m, but this double sum, this is a series of positive terms. So, it is equal to the repeated some 1 to infinity and E n intersection A m.

And each of these is m E n. So, this is equal to the sun of n E s. So, this proves that the countable additivity holds for any disjoint collection of Lebesgue measurable subsets of R d.

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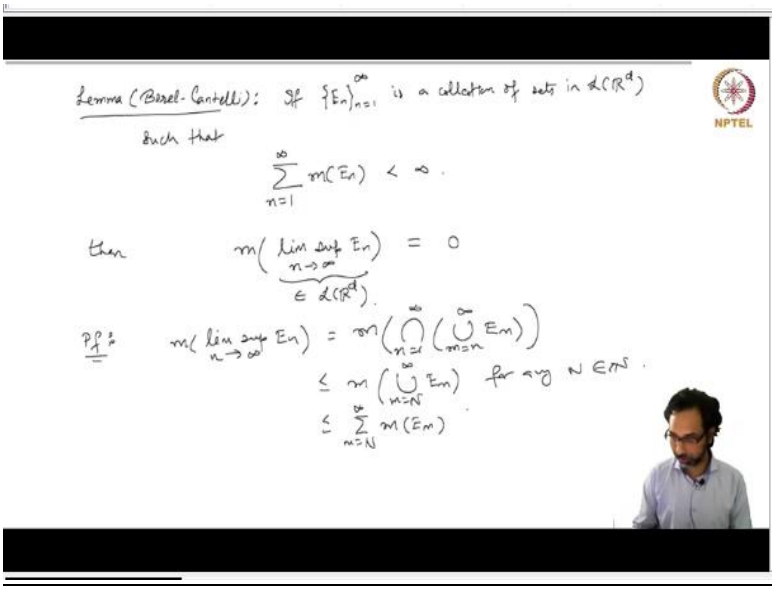

Now, let me conclude this lecture with a very useful lemma called the Borel-Cantelli Lemma. And it says that if E n n equal to 1 to infinity is a collection of sets in L R d, meaning that each E n is Lebesgue measurable such that the sum n equal to 1 to infinity m E n is finite. So, the series converges then the measure of the lim sup of E n, n goes to infinity, this measure is 0. So, we know that this is a measurable set, this is a Lebesgue measurable set, this right belongs to L R d and the measure of this measurable set is 0. So, let us see the short proof. So, we know that, the measure the lim sup of E n as n goes to infinity.

This is the measure of the intersection n equal to 1 to infinity union m equal to n to infinity of E n and so, this is less than or equal to the sum. Well, first, it is less than or equal to m measure of m equal to capital N to infinity E n for any n belong to n and so this is less than or equal to the sum from m equal to capital N to infinity m E m. So, now we will use the Cauchy criterion for the convergence of series.

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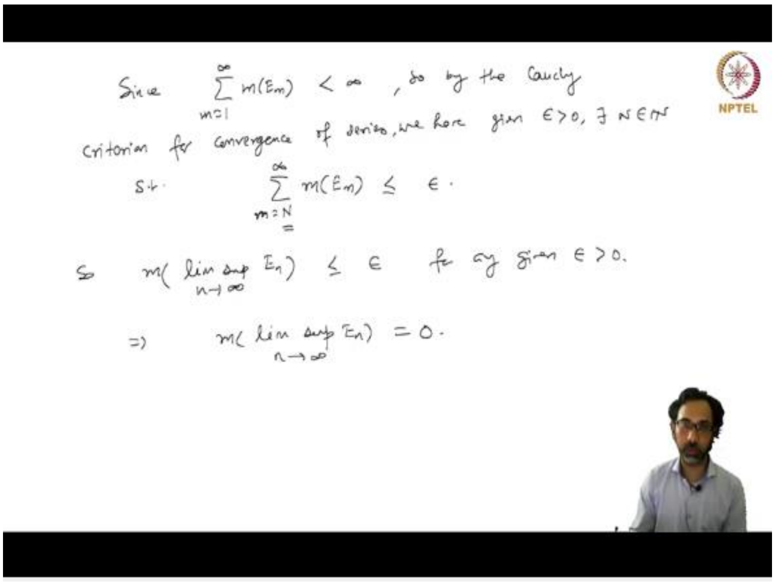

So; since the sum m equal to 1 to infinity and measure of E n. This is a finite sum. So, by the Cauchy criterion for convergence of series, we have given epsilon greater than 0 there exists n a natural number such that the sum n equal to n to infinity, so, the tail of the series is less than or equal to epsilon. So, you can choose n high enough**.** So, that the tail of the series becomes smaller and smaller.

And so, the measure of the Lim sup as n goes to infinity E n is less than or equal to epsilon for any given epsilon greater than 0, because, here we can choose n from the Cauchy criterion and you will have this inequality. So, of course, this implies that the lim sup is 0, because it is arbitrarily small, this must be arbitrarily small non negative number this must be 0.