Measure Theory
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Lecture - 26
Equivalent Criteria for Lebesgue Measurability of a Subset Part – 2

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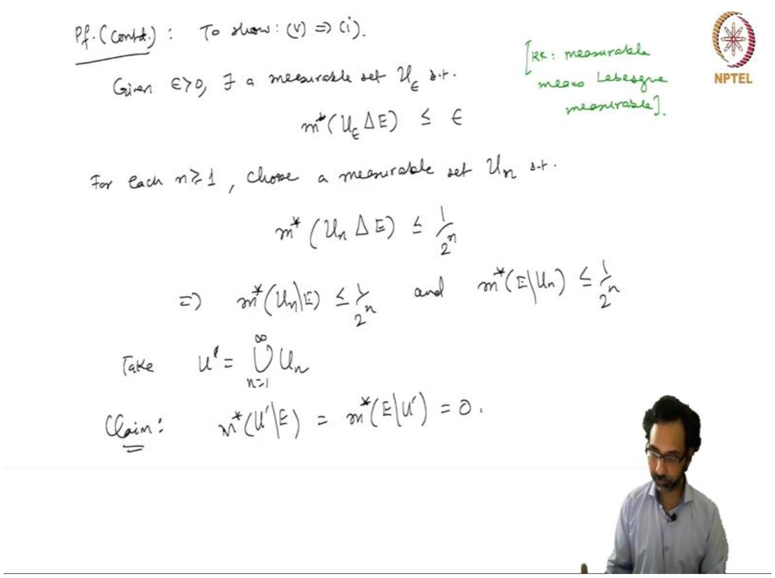

So continuing the proof so this is to show that 5 implies 1. So given epsilon greater than 0 the condition in 5 was that it is almost measurable. So there exists a measurable set let me write it as u epsilon such that the outer measure of u epsilon symmetric difference with E is less than or equal to epsilon. So what we can do is for each n greater than or equal to 1 choose measurable set. So measurable here always means measurable means lebesgue measurable.

Because you are only dealing with lebesgue measurement sets for now later on we will come to other measurable measures and other kinds of measurable sets but here by measurable set we mean a lebesgue measurable set. So for each n greater than or equal to 1 we choose a measurable set you u n such that the outer measure of u n symmetric difference with E is less than or equal to 1 over 2 to the power n.

So we are choosing for each n epsilon to be 1 / 2 to the power n and we can choose for that epsilon a measurable set which we denote by u n such that the symmetric difference has outer measures less than or equal to 1 / 2 to the power n. So this implies that the outer measure of u n -

E is less than or equal to 1 over 2 to the power n and the outer measure of E – u n is less than or equal to 1 over 2 to the power n. So now we take the union of all these u n’s which we denote by u prime. So take the union n = 1 infinity of this u n's and I claim that the outer measure of u prime - E is equal to the outer measure of E - u prime and both are equal to 0. So let us try to show this.

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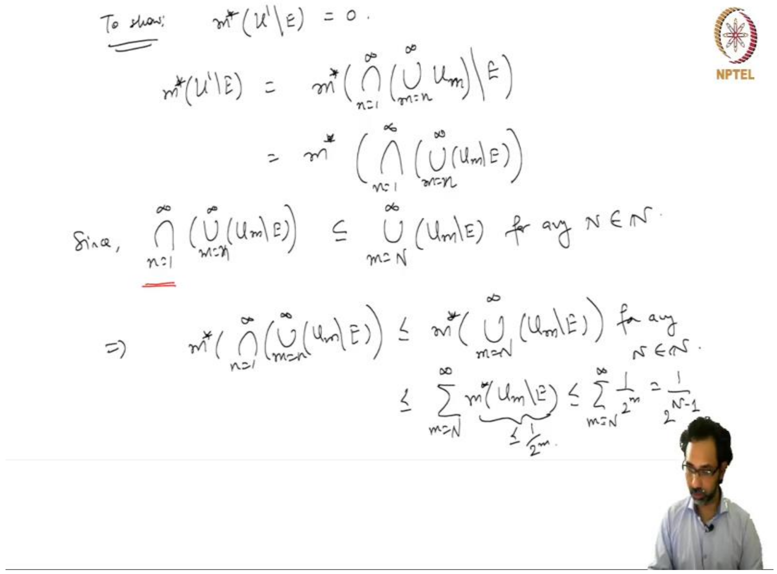

So to show that the outer measure of let us say u prime – E = 0, now u prime – E = m star of so u prime was the lim sup so we had the intersection n = 1 to infinity of the union m = n to infinity u m – E and this is equal to m star of the intersection of all the sets union m = n to infinity u m - E and now note that the intersection since the intersection of all these u m - E is a subset of m = n to infinity u m - E for any natural number and because we are taking the intersection here of all n.

So if you fix any N then this intersection is a subset of the part of the lim sup that starts at m = N. So this implies that m star of the intersection u m = n to infinity u m - E is less than or equal to the m star of m = N to infinity u m - E for any N belonging to N and now the right hand side is less than or equal to the sum m = N to infinity m star u m - E and each of them is less than or equal to 1 over 2 to the power m. So this whole sum is less than or equal to the sum from N to infinity 1 over 2 to the power m which is equal to 1 / 2 to the power N - 1.

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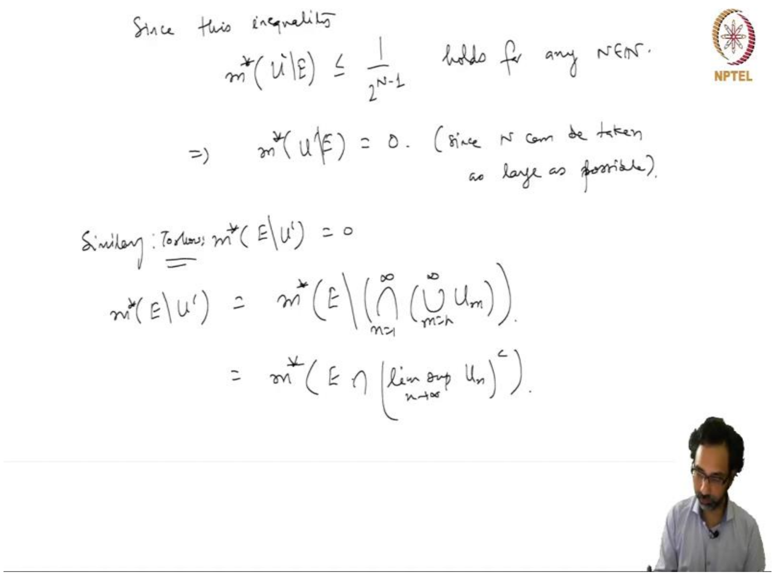

So this is true for since this is true for any N this inequality which says that m star of u prime - E is less than or equal to 1 over 2 to the power N - 1 holds for any N this implies that m star of u prime - E = 0 because you can have N as large as possible since N can be taken as large as possible. So m star of u prime – E can be as small as possible and so in fact it is 0. Similarly to show that m star of E - u prime is also 0.

So this is to show again, so now we have m star of E - u prime this is equal to m star of E minus I will just write down the definition of lim sup again m = n to infinity u m and this is equal to m star of E intersection with the compliment of the lim sup. So now we can rewrite this.

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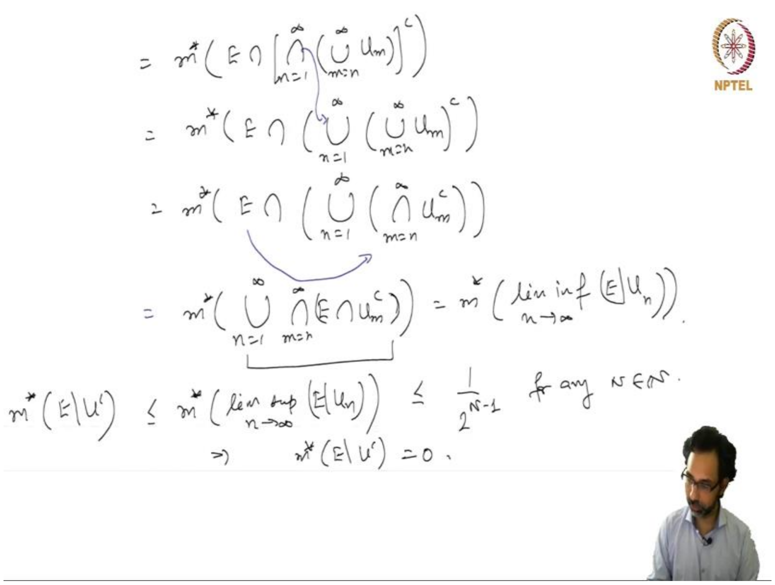

m star E intersection so the compliment n = 1 to infinity union m = n to infinity u m whole complement this is equal to m star E intersection union so I am going to use the Morgan's law. So union n = 1 to infinity because the intersection changes to union because we have a compliment and inside we have union m = n to infinity u m compliment. So this I can again be rewritten as E intersection union n = 1 to infinity.

And now I am again going to use them Morgan's laws. So you get m = n to infinity u m compliment. So now we can take this E inside the union of these sets. So you get m star union n = 1 to infinity E intersection m = n to infinity E intersection u m compliment. So this is nothing but this set this whole set is nothing but the lim inf as n goes to infinity of these sets E - u n and because the lim inf is contained in lim sup we have an inequality lim sup E - u n.

And again we can use a similar argument is above to show that this is again bounded above by 2 to the power n - 1 for any N belongs to N. So in the end we have on the left hand side E - u prime which is bounded above by 1 over 1 to the power n - 1 for any n belong to the natural number, so as large as possible. So this implies that E - u prime = 0. So we have shown that both E- u prime and u prime - E. So we have proved this claim that both u prime - E and E - u prime have after measures 0.

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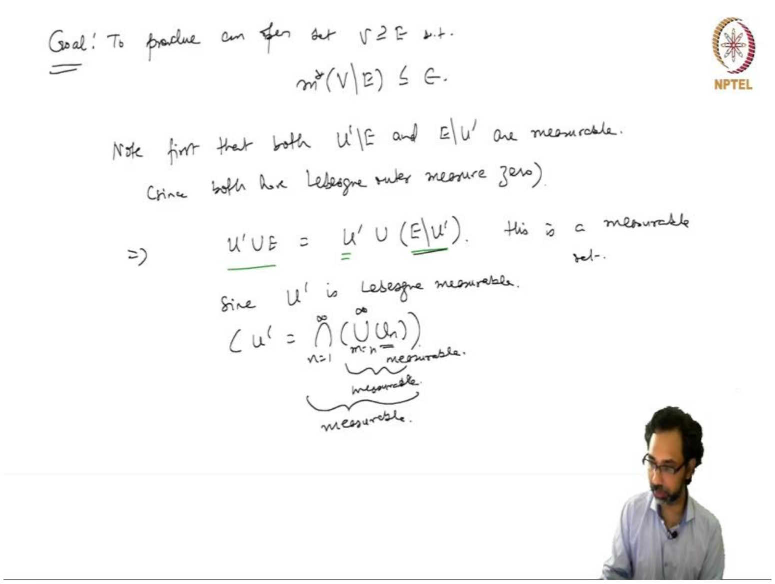

Now how to use this fact to produce an open set now we have to produce our goal is to produce an open set. So this is what you want to prove the first item because we are starting from 5 and then we are trying to prove 1. So this is the first item is to produce an open set V containing E such that the outer measure of V - E is less than or equal to epsilon. So to do this note first that both u prime - E and E - u prime are measurable since both are out both have lebesgue outer measures 0.

So we have already seen that such sets are lebesgue measurable. So this implies that u prime union E which is now can which can be written as u prime union E - u prime this is a measurable set because since u prime is lebesgue measurable remember that u prime was the intersection of the union of the u n’s. So each of these u n’s are measurable which means that this countable union for each n is measurable, which means that this intersection is also measurable.

So from our class of lebesgue measurable sets we see that u prime is lebesgue measurable and E - u prime is also lebesgue measurable, so both are lebesgue measurable sets. So the union u prime union E is lebesgue measurable set.

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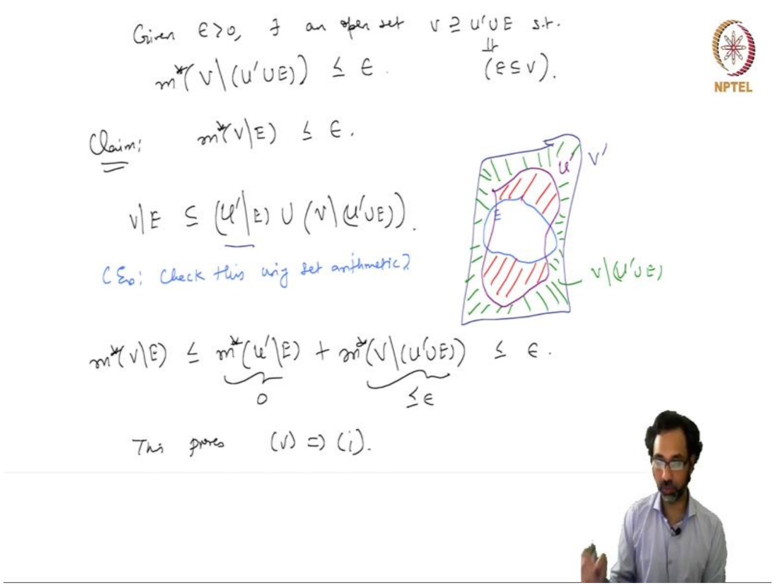

So given epsilon greater than 0 there exists an open set V containing u prime union E such that m star of V - u prime union E is less than or equal to epsilon. Now because V is a superset of u prime union E this also means that this also implies that E is a subset of V. So I claim that this is our required open set and m star of V - E is less than or equal to epsilon. So let us draw a venn diagram of what is happening here.

So you have E and you have another set u and you have the set V containing the union of u and E. So I claim that V - E is contained inside u prime. So this is u prime, u prime - E union V - u prime union E. So let us see what is u prime - E. So this is this part this is u prime - E and V - u prime union E this is given by this part which lies outside u prime as well as outside E. So all this part in the green shaded region is this is V - u prime union E.

So in fact it is not very difficult to show that V - E which is kind of the union of the red and green parts is a subset of the union of the red and green parts. So one can just see it visually but again as an exercise check this using set arithmetic. So now we are done because m star of V - E is less than or equal to m star of u prime - E + m star of V - u prime union E this is 0 and this is less than or equal to epsilon, so this is less than or equal to epsilon. So we have found an open set that contains E and m star of V - E is less than or equal to epsilon. So this proves that 5 implies 1.