Measure Theory
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Lecture - 25
Equivalent Criteria for Lebesgue Measurability of a Subset - Part I

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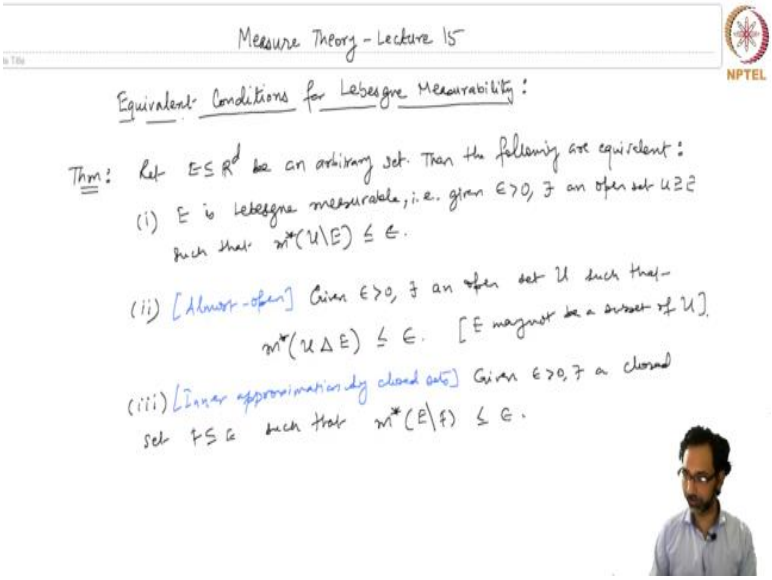

In the last lecture, we have seen that the class of Lebesgue measurable sets is a, is very big and in particular it can it is closed under taking complements, taking countable unions and taking countable intersections. So, in this lecture, we will look at some equivalent criteria for Lebesgue measurability we have defined, we have given 1 definition of Lebesgue measurability and we will see that there are plenty of other equivalent criteria that one can write to for the definition of Lebesgue measurable sets.

So, I have stated this as a theorem let E be an arbitrary subset of R d, an arbitrary set, then the following are equivalent. So, the first is E is Lebesgue measurable of course, we want to give equivalent conditions for Lebesgue measurability. So, the first one is basically just the definition of Lebesgue measurability, this is 1, that exists given epsilon greater than 0 there exists.

So, let me fix given any epsilon greater than 0 there exists an open set u containing E such that the outer measure of the compliment u set compliment u - E is equal to is less than or equal to epsilon. The second one is it can be termed as almost open criteria which says that given epsilon greater than 0, there exists an open set u now, E may not necessarily be contained in u, but nevertheless we have that the rather than the set difference, we will take the symmetric difference u symmetric difference E is less than or equal to epsilon.

So, here the difference between 1 is that E may not be a subset of u. The third is approximation by closed sets inner approximation by closed sets. So, this criteria says that given epsilon greater than 0, there exists a closed set F contained in E, this is an inner approximation. So, F is contained inside E such that the outer measure of E - F is less than or equal to epsilon.

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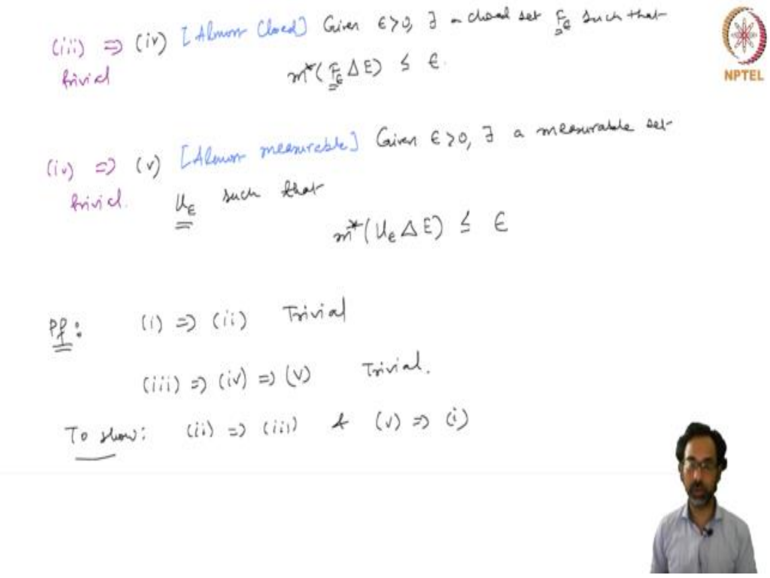

The 4th condition is similar to the almost open condition; this is the almost closed condition. It says that given epsilon greater than 0, there exists a closed set F. Now, F need not be contained inside E that is the only difference between this case and the third case, such that outer measure of F symmetric difference E is less than or equal to epsilon and the fifth one, this is almost measurable condition this says that given epsilon greater than 0, there exists a measurable subset.

Measurable set E epsilon such that the outer measure of let me write it, u epsilon symmetric difference with E is less than or equal to epsilon. So, here all these sets this set and this closed set in the last condition all of them depend on epsilon. So, let me put an epsilon everywhere, this is u epsilon, u epsilon, u epsilon, F epsilon. So, these are 5 equivalent conditions so, we have listed 5 equivalent conditions for the, if you want to test some whether some given set is Lebesgue measurable or not so, let us try to show this so, few of these implications.

We will start with 1 and we will see that few of these implications are quite trivial. For example, 1 implies 2 is trivial, this is trivial, because, if E is a, if you have an open set u epsilon containing E, then they that will also satisfy this condition here, because if it satisfies this condition, then this condition is the second condition is trivial similarly, 3 implies 4 is trivial. So, 3 implies 4 this is also trivial and 4 implies 5 is also trivial because, if you can find a closed set, then a closed set is a measurable set.

So, fifth one is automatically satisfied, if 4 is satisfied. So, 1 implies 2 is trivial, so, let me know down these things this remarks. So, 1 implies 2 is trivial then, then we have 3 implies 4 is trivial, 3 implies 4 implies 5 these are also trivial. So, one just to has to show that 2 implies 3 and 5 implies 1 then we will be done.

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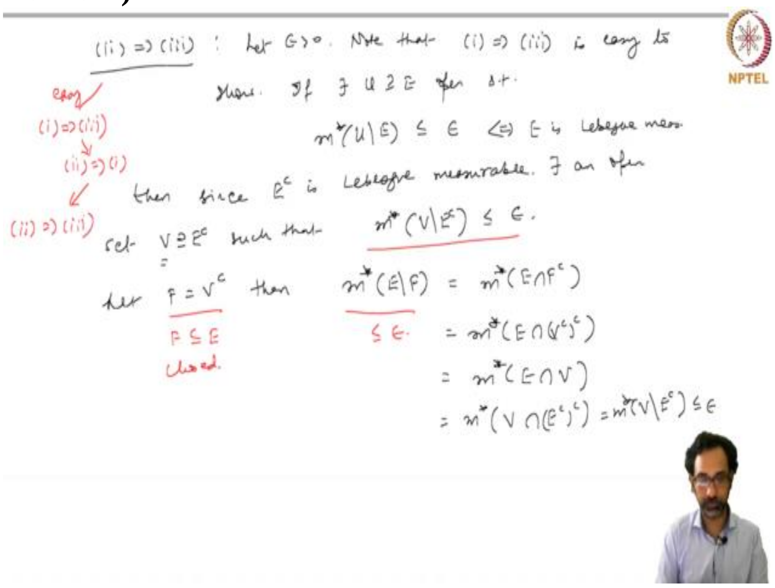

So, let us try to show whether 2 implies 3. So, let epsilon greater than 0 be given. Now, note that 1 implies 3, 3 is quite easy to show, because, if we have an open set u containing E such that m star u - E is less than or equal to epsilon then since, so, this implies that E is Lebesgue measurable, then this also implies that E complement is Lebesgue measurable this we have shown before that if E is Lebesgue measurable then, so is E compliment.

So, then there exists an open set V containing E compliment such that, m star of V - E compliment is less than or equal to epsilon. Now, we have if we want to prove 3, which is to given a subset a closed subset of E such that auto measure of E - F is less than or equal to epsilon, then we can take F to be the complement of this set V let F be V compliment, then the outer measure of E - F, is this is equal to m star E intersection F complement which is equal to m star E intersection V complement, complement.

But this is nothing but E intersection V but we can also write it as V intersection E complement complement and this is nothing but m star of V - E complement and this is less than or equal to epsilon. So, this from the Lebesgue measurability of the compliment, we deduce a set F lying inside E. So, F is lying inside E and F is closed such that m star of E - F is less than or equal to epsilon.

So, to 1 implies 3 is easy so, let us show that 2 implies 1 and then that would imply that in turn 2 implies 3. So, we have broken up this into 2. The first one is 1 implies 3 this is easy and we will also prove that 1, 2 implies 1 and so, this will imply that 2 implies 3.

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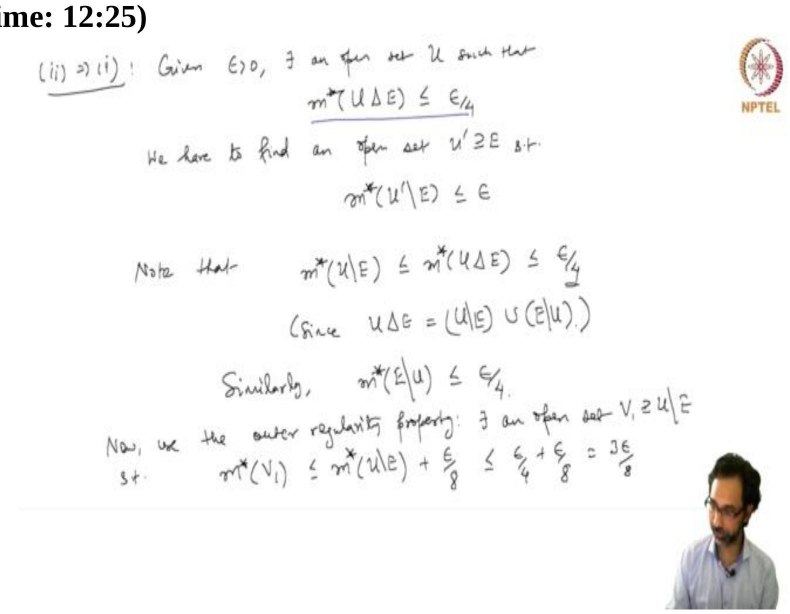

So, let us try to show that 2 implies 1 which means that we are given, so, this is 2 implies 1 given epsilon greater than 0 there exists an open set u such that there the outer measure of u symmetric difference with E is less than or equal to epsilon. Now, u may not contain E as a subset. So, we have to find an open set u prime that contains E in this case such that the outer measure of u prime - E is less than or equal to epsilon.

So, to produce such an open set u prime we will use what we already know from this inequality. So, let me take epsilon / 4 here and you will see why I am taking epsilon / 4. So, note first that note that m star of u - E is less than or equal to m star of u symmetric difference E which is less than or equal to epsilon / 4 because, since, the symmetric difference is the union of u - E and E - u. Similarly, m star of E - u is also less than or equal to epsilon / 4. So, now, we can use the outer regularity property of the Lebesgue outer measure.

So, this implies that there exists an open set V 1 that contains u - E such that m star of V 1 is less than or equal to m star of u - E plus, I am going to take epsilon / 8 here. So, that this is bounded above by epsilon / 4 + epsilon / 8 and this is equal to 3 epsilon / 8.

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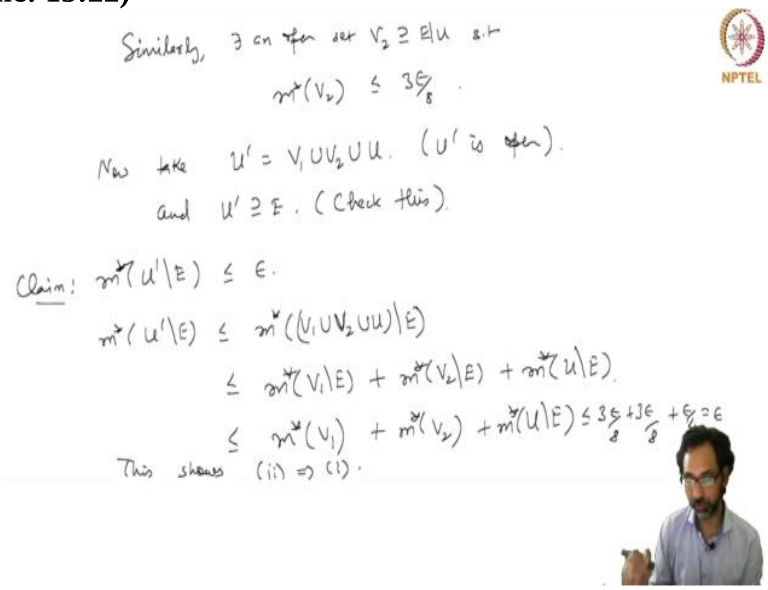

Similarly, there exists an open set V 2 containing E - u says that the outer measure of V 2 is less than or equal to 3 epsilon / 8. Now take u prime to be the union V 1 union V 2, union u, so it is a union of 3 open sets. So, it is open u prime is open and u prime contains E as well, u prime contains E. So check this now, I want to show that m star of u prime - E, I am claiming that m star of u prime - E is less than or equal to epsilon so, let us see why?

So, m star of u prime - E because this is less than or equal to m start of V 1 union V 2 union u - epsilon - E and this is less than or equal to m star of V 1 - E + m star of V 2 - E + m star of u - E. But the first 2 terms are bounded above by m star of V 1 + m star of V 2 and then you have m star of u - E and so, this is less than or equal to 3 epsilon / 8 + 3 epsilon / 8 + epsilon / 4 which is equal to epsilon.

So, we have shown that u prime is a, is an open set that contains E and the outer measures of u prime - E is bounded by epsilon. So, this shows that part 2 implies part 1. So, going back to the beginning of the proof, we had 2 things that we needed to show the first one was that 2 implies 3. So, this we did and now, we have to show that 5 implies 1. So, recall that 5 is the almost measurable condition which says that given epsilon greater than 0.

There exists a measurable set u epsilon such that the symmetric difference of epsilon with E is less than or equal to epsilon and then we have to produce an open set that contains E such that then outer measure of that open set minus E is less than or equal to epsilon. So, this would be 5 implies 1. So, let us try to show that 5 implies 1 for this purpose, we will have to define a couple of new notions.

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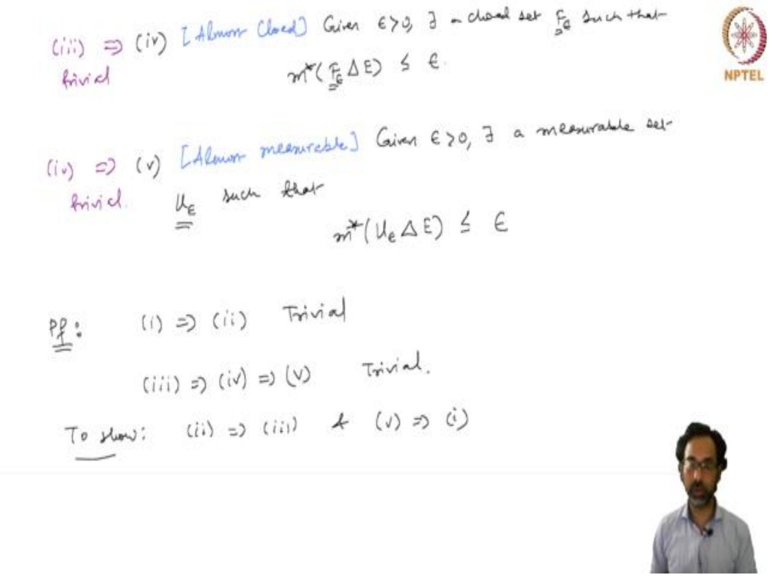

So, let me define the notions of limit superior and limit inferior of sets in R d. So, let A n, n = 1 to infinity be a sequence of subsets of R d then they may not be subsets of R d, this is a very general notion, but since we are only dealing with subset of R d for the moment so I have taken them to be subsets of R d. Then we define the lim sup the limit superior as n goes to infinity of A n this is by definition the intersection n = 1 to infinity of the union m greater than equal to n.

So, m from n to infinity A m and similarly, the limit inferior lim inf of A n as n goes to infinity, this is the union n = 1 to infinity and then we take the intersection of m = n to infinity A m. So, to gain an idea of what these lim sup and lim inf describe, the following lemma is quite useful it says that the lim inf of A n as n goes to infinity, this set can be described as the set of all x in R d such that x belongs to all but finitely many A n's.

So, all but finitely many meaning that there may be finitely many A n’s say n 1, n 2 and n k for which x does not belong to those sets, but it belongs to all the other sets. Similarly, the lim sup of A n as n goes to infinity this is the set of all points says that x belongs to infinitely many A n’s. So, the difference between the 2 is that x for the second one x may not belong to all but finitely many.

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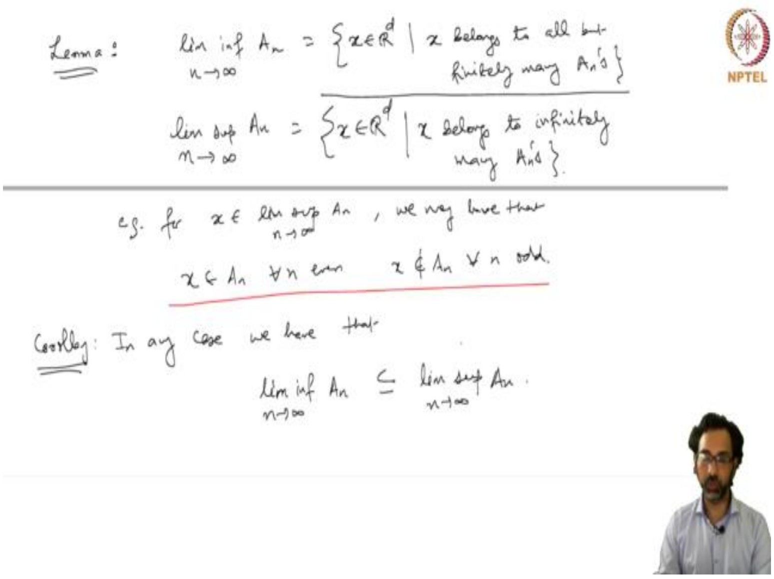

For example, eg for x in lim sup of A n we may have that x belongs to A n for all n even, and x does not belong to A n for all n odd. So, in this case, there are infinitely many sets A n such that x does not belong to those sets. So, this x does not belong to the lim inf but it belongs to the lim sup, but in any case, we have that lim inf have A n is a subset of lim sup of n. So, this is a corollary of the lemma that because, this set is defined.

If we take this definition as the set of all x which belongs to all but finitely many and the right hand side is belongs to infinitely many. So, of course, if x belongs to all but finitely many A n’s, then it belongs to infinitely many A n’s, but the vice versa is the converse is not true as we see in this example. So, in this way, we have a more tangible definition or more practical definition of what lim sup and lim inf actually describe. Now, we will use these 2 sets to prove our implication from 5 to 1.