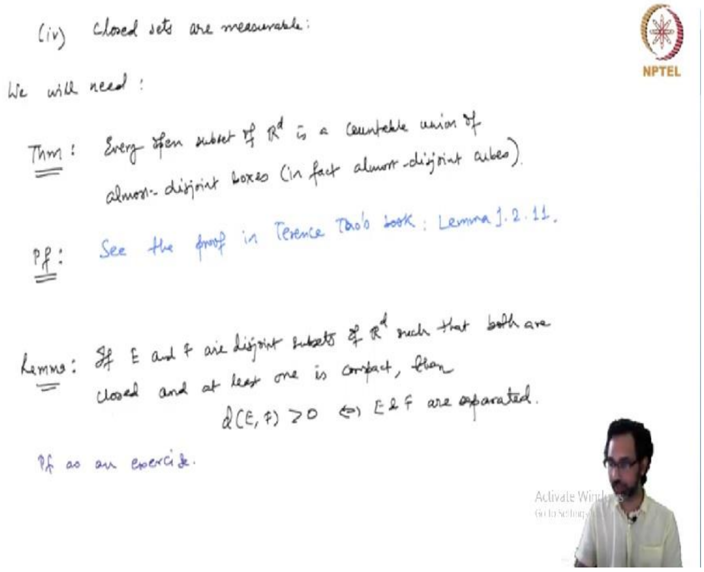
Measure Theory
Prof. Indrava Roy
Department of Mathematics
Institute of Mathematical Science

Lecture - 24
Lebesgue measurable class of sets and their Properties Part 2

(Refer Slide Time: 00:17)



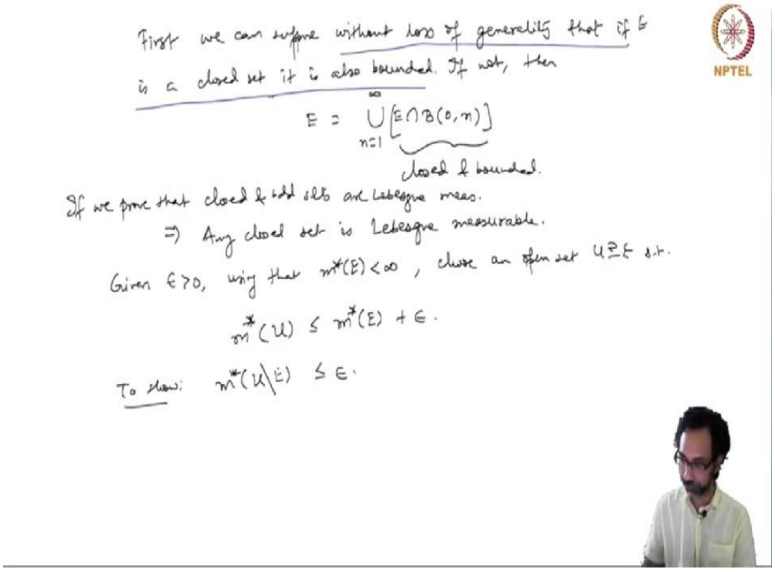
Now, let us show the fourth part which was that closed sets are lebesgue measurable. And to prove this result, so this fourth part is requires some more careful attention than the other parts. And so, we will need 2 results to prove this fourth part. So first is a theorem, it says that every open subset of R d is a countable union of almost disjoint boxes. In fact, almost disjoint closed cubes.

So, I will not prove this theorem and I will just refer you to see the proof in Terence Taos book. And this is, it is a given in lemma 1.2.11. So, the proof is quite detailed there and not very hard to follow. So, I will leave the proof for you to learn and understand. Of course, if you have any questions you can ask on the portal and we will be happy to answer. So, this is the first result that we will need that every open subset of R d can be written as an union of countable union of almost disjoint boxes.

The second theorem that we need or rather this is a lemma is that this is a topological fact that if E and F are disjoint subsets of R d such that both are closed and at least one is compact then the distance between E and F is greater than 0 which is to say that E and F are separated.

So, I will again leave this as an exercise. This is an exercise in topology which uses the concept of compactness and so we will use these two results to prove that close sets are measurable. So, let us see how this is done.

(Refer Slide Time: 03:53)

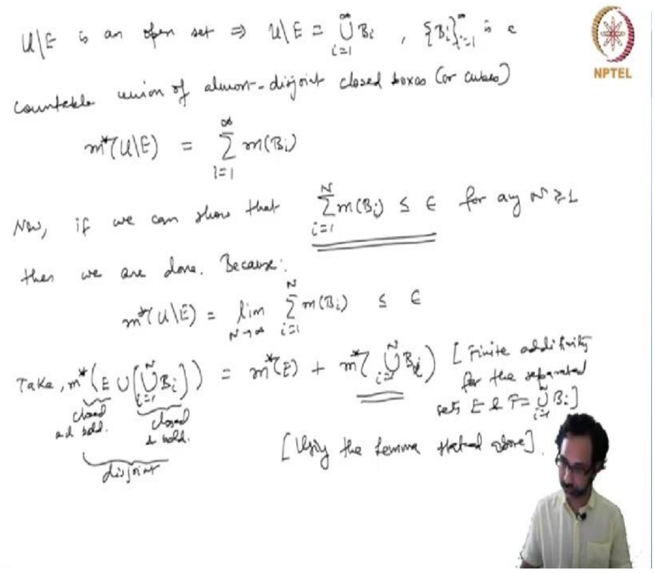


So, first we can suppose without loss of generality that if E is a closed set it is bounded also, it is also bounded. Because, if not then E can be written as the union n = 1 to infinity E intersections with let us say the ball with centre 0 and radius n. So, now this is a closed and bounded set and so if we prove that closed and bounded sets are lebesgue measurable then, E will be a countable union of closed and bounded sets which are each of them are lebesgue measurable. So, therefore this will imply that.

So, if we prove that closed and bounded sets are Lebesgue measurable, then this will imply that any closed set is in fact, Lebesgue measurable. Because we can write it as a countable union of closed and bounded sets. So, we can suppose, without loss of generality that E is a closed set which is also bounded. So, now we can use the, now see given epsilon greater than 0 and using the fact that m star E is finite because now it is closed and bounded also.

So, the lebesgue outer measure is going to be finite. Choose an open set u containing E says that we have m star u is less than or equal to m star E + epsilon. So, we have to show that m star of u - E is less than or equal to epsilon.

(Refer Slide Time: 06:46)

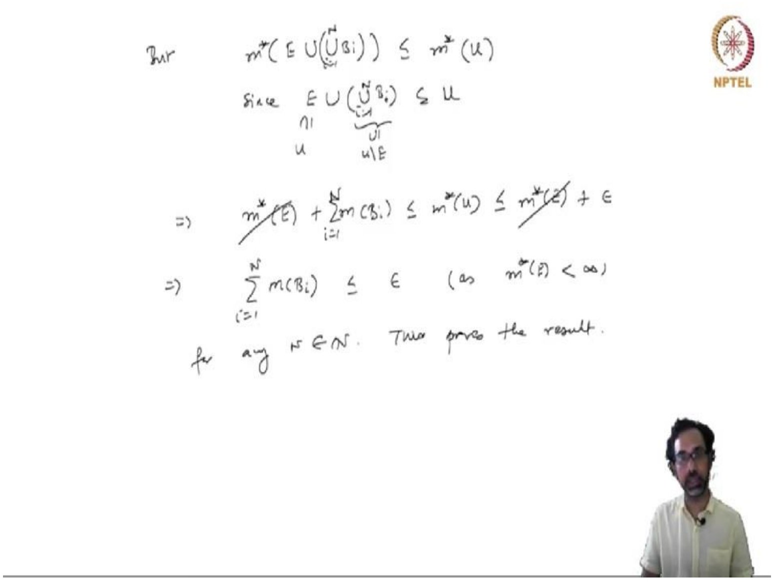


So, note that u - E is an open set. So, this implies that u - E can be written as a countable union of almost disjoint boxes. So, B i is a countable union of almost disjoint closed boxes or cubes. This is from the theorem that we stated above. And so, we get that m star u - E is the sum of all these, almost disjoint closed boxes 1 to infinity. Now, if we can prove we can show that this sum i = 1 to N m B i is less than or equal to epsilon for any N greater than or equal to 1.

Then we are done because what we have then this that m star u - E is equal to the limit of N tends to infinity i = l to 1 to N m of B i and since each of them is less than or equal to epsilon the limit is also less than or equal to epsilon. So, we have to show that any finite sum i = 1 to N m B i is less than or equal to epsilon. Now, take for this purpose we can take E union of this N closed boxes. So, here E is closed and bounded and these are also closed and bounded. So, this implies and also that this is a disjoint union.

So, this implies that m star of the union of these 2 things is equal to m star E + m star union i = 1 to N B i because of finite additivity for the separated sets E and F which is the finite union of the B i. And here we are using the lemma stated above. So, m star E union this finite union of B i = m star E + m star of this union of B i from i from 1 to N.

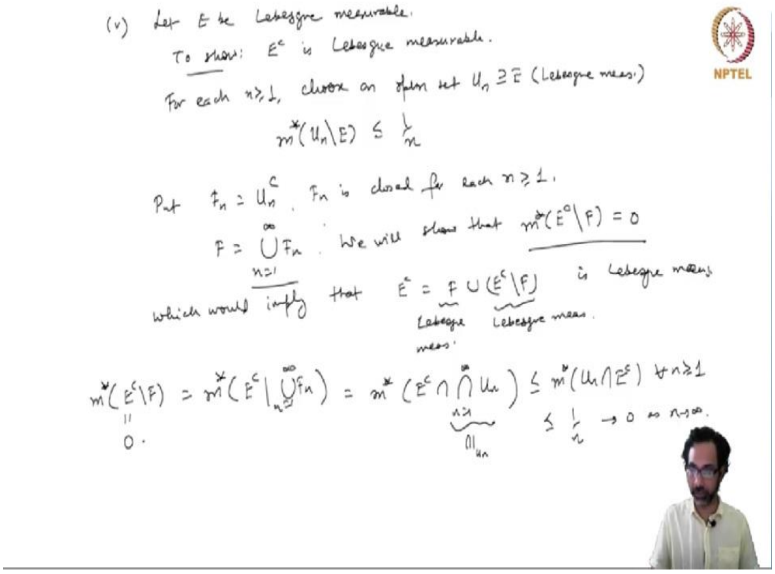
(Refer Slide Time: 10:54)



But, m star of E union, union 1 to N B i this is less than or equal to m star of u because this set is a subset since E union B i is a subset of this set u because this is a subset of u and this is a subset of u - E. So, the union is still a subset of u. So, by monotonicity we get this inequality and on the left hand side we have m star E plus some of m B i, i from 1 to N. And on the right side we get m star u but remember that u was chosen so that it was less than or equal to m star E + epsilon.

So, this implies that i = 1 to N m B i is less than or equal to epsilon, since m star of E is finite, so we can cancel out this m star E and this m star E. And so, we got what we needed which is that this is this sum i = 1 to N m B i is less than or equal to epsilon for any N. So, this proves the result. So, we have proved that closed set are also lebesgue measurable.

(Refer Slide Time: 12:56)



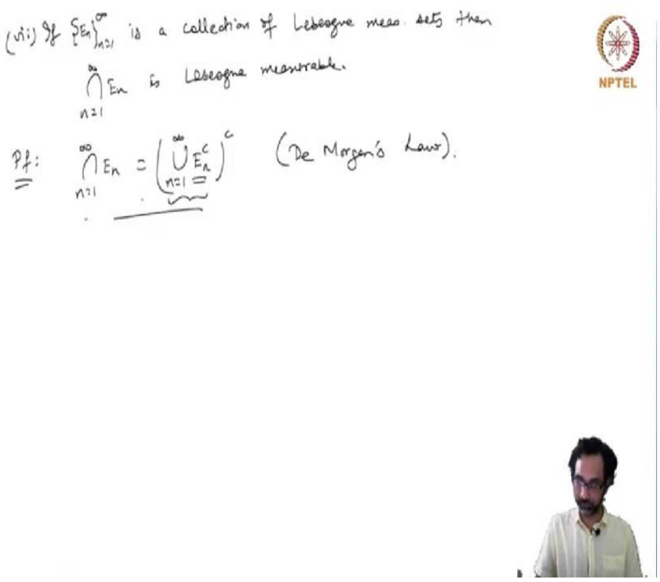
Now, let us come to the compliment in this was part 5. So, let E be lebesgue measurable and we have to show that, E compliment is lebesgue measurable. So, we proceed as follows for each natural number n greater than equal to 1, choose an open set u n containing E, this is a lebesgue measurable set. So, we can choose u n such that m star of u n - E is less than or equal to 1 / n. So, here we are taking epsilon to be 1 / n and for each n we are choosing an open set u n such that this inequality is satisfied. And now, put f n = u n compliment.

So, we would like to take, so this is a closed set. F n is closed for each n greater than or equal to 1 and now, we put F equals the union of all these F n is from n = 1 to infinity. So, we will show that m star of E compliment - F = 0 which would imply that E compliment which is the union of F and E compliment - F and now, if this measure is after measure of E compliment - F is 0, then this is lebesgue measurable and F being a countable union of closed sets, this is Lebesgue measurable.

So, the union is E complement is lebesgue measurable. So, we have to show that m star of E compliment - F = 0. So, how do we show this. So, note that m star of E compliment - F = m star E compliment minus the union n = 1 to infinity. This is the by definition of F here and so, this is m star E compliment intersection with the intersection of the complements of F n. So, each F n complement is u n by definition.

So, this is less than or equal to m star u n intersection E complement for all n greater than equal to 1, because this is contained in u n for each n. And we have chosen this such that, this is 1 over n is less than or equal to 1 over n. So, this is true for all n, therefore this goes to 0 as n tends to infinity. So, therefore, this m star of E compliment - F is 0 and we are done. So, if E is lebesgue measurable, then E complement is also lebesgue measurable.

(Refer Slide Time: 17:15)



Finally, we come to the last part which is the 7th part which says that if E n 1 to infinity is a collection of lebesgue measurable sets, then the intersection n = 1 to infinity E n is lebesgue measurable. So, this is quite easy. So, one can write intersection of these E n as the union n = 1 to infinity E n compliment. So, this is by De Morgans laws. So, since each E n is lebesgue measurable, this E n compliment is lebesgue measurable.

So, the countable union of lebesgue measurable sets of E n compliment or lebesgue measurable and finally, you take the compliment which is again lebesgue measurable. So this is just a one line proof that countable intersections are also lebesgue measurable.