Measure Theory
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Lecture – 23
Lebesgue measurable class of sets and their properties

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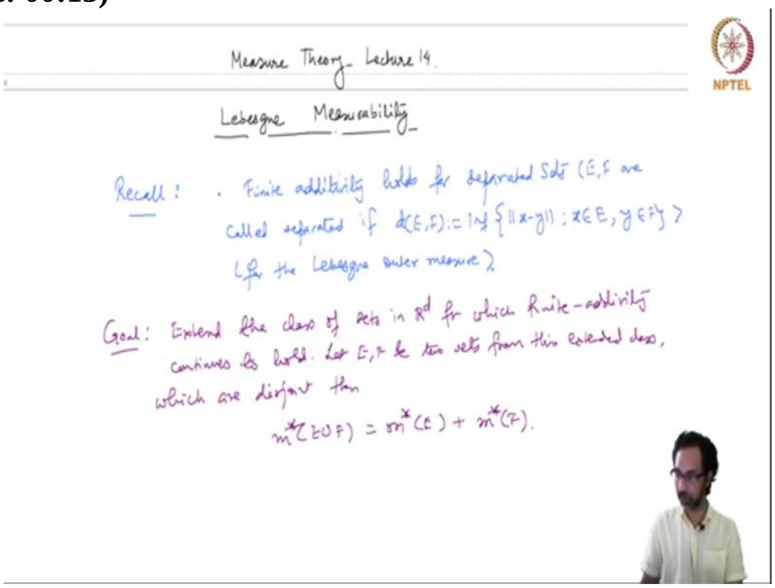

Now, we come to a very important concept and this is called Lebesgue measurability. So, this will allow us to define what are called Lebesgue measurable sets. And if you recall from one of the first videos in the first week, we have mentioned that due to the banach tarski paradox, we cannot expect finite additivity to hold for any 2 arbitrary disjoint sets. And therefore, we need to categorize, the sets of subsets of R d into 2 classes, one will be called measurable and one we will call non measurable and the measureable sets will be expected to satisfy the finite additivity rule.

Now, recall from the last lecture, that we have finite additivity holds for what are called separated sets. So, remember that E and F are called separated if the distance between these 2 sets, which is defined as the infimum of all Euclidean distances says that x belongs to E and y belongs to F this is strictly greater than 0 . So, this is a definition for separated sets and we have seen that finite additivity for the living outer measure holds for separated sets.

So, this is for the Lebesgue after measure. So, now, the goal is to extend the class of sets in R d for which finite additivity continues to hold which means that if you take 2 sets in this class, which are disjoint then So, let us say let E and F be 2 sets, 2 sets from this extended class which we still have to define which are disjoint then, the Lebesgue outer measure of E union F should be equal to the living outer measure of E plus the Lebesgue outer measure of F and we will see that for what we call the Lebesgue measurable sets this will be true.

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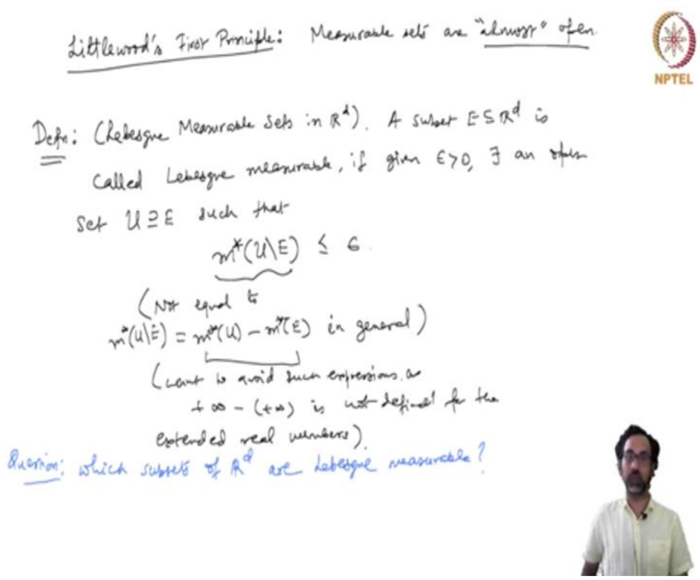

So, the idea is to follow what is called the Littlewood's first principle. So, there are in total 3 principles of Littlewood's in analysis that we will encounter. So, the first one says that measureable sets So, these are the sets in this class which will follow finite additivity are almost open, so, I am putting this in quotes and I will make this precise here, but the idea is to approximate measurable sets with open sets.

So, this may class of measurable sets are called almost open. So, this is Littlewood's first principle. So, now we can define following Littlewood's first principle. What are the Lebesgue measurable sets in R d So, a subset E of R d is called Lebesgue measurable if given epsilon greater than 0, there exists an open set, because we wanted to approximate a measurable set by open sets, which is why this is almost open.

So, there exists an open set U, which contains E such that the Lebesgue outer measure of the set compliment of u - E is less than or equal to epsilon. So, here notice that I am not writing this is not equal to m star u - m star E in general, and not only that, we would like to avoid such expressions, because, even in the extended real numbers plus infinity -, + infinity is not defined. So, this is for the extended real numbers.

So, we want to avoid such things, but for nice subsets we will see that is this equality m star u - E = m star u - m psi this will hold but, we have to be careful in how we define this. So, once we have the notion of what is the Lebesgue measurable set then of course, the natural question is to ask is which sets are which subsets of R d are Lebesgue measurable? So, we will answer this question in this lecture today and we will see that this is a quite a large class of sets, which is Lebesgue measurable.

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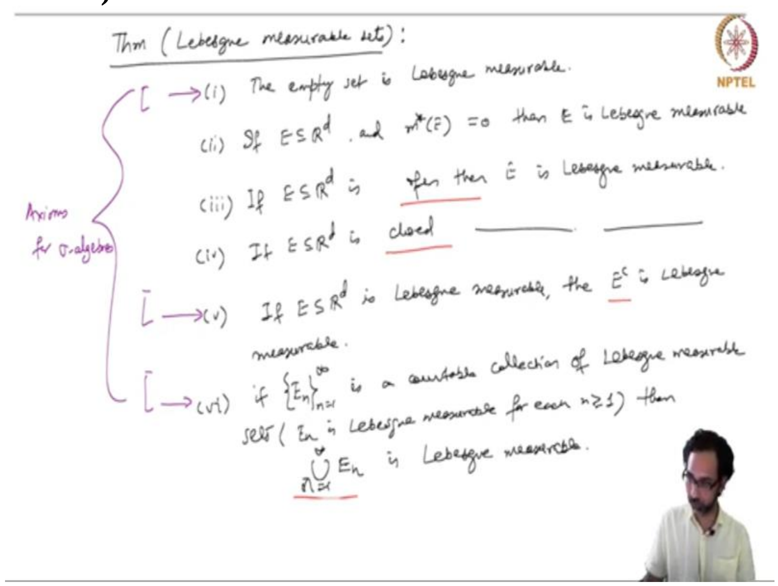

So, our next theorem lists what are the primary examples basic examples of Lesbegue measurable sets. So, the first one is that the empty set is Lebesgue measurable. The second is that if E is a subset of R d and the outer measure is 0 then E is Lebesgue measurable. The third is if E is an open set then E is Lebesgue measurable, fourth is that if E is closed is this is open this is closed then E is Lebesgue measurable.

The fifth property the fifth class of sets which are Lebesgue measurable is given by a finite countable union of Lebesgue measurable sets. So, the fifth one is that if E is it is a Lebesgue measurable then E complement is Lebesgue measurable. Sixth property is that, if E n, n = 1 to infinity is a countable collection of Lebesgue measurable sets which means that E n is a

Lebesgue measurable for each n greater than or equal to 1 then the union n = 1 to infinity E n is Lebesgue measurable.

And lastly, the seventh property is that if E n, n equals 1 to infinity is a collection of Lebesgue measurable sets then the intersection n = 1 to infinity E n is again Lebesgue measurable. So, we see that this already covers a large variety of subsets of R d and in particular, every open set every closed set is Lebesgue measurable a complement of Lebesgue measurable set is Lebesgue measurable.

And Lebesgue measurable sets are closed under countable unions and countable intersections meaning that countable union of the Lebesgue measurable sets is measurable and countable intersections of Lebesgue measurable sets are Lebesgue measurable. So, just remark that the properties 5, 6 and 7 constitute, let me remark that the property the first property, the fifth property and the sixth property these 3 properties are together constitute the axioms for what are called sigma algebras.

So, we will see abstract sigma algebra later, but in terms of sigma algebra is the these 3 property says that Lebesgue measurable sets former sigma algebra, so, the empty set is included complement of a set in the algebra is included as well as countable unions of measurable sets are included within the algebra.

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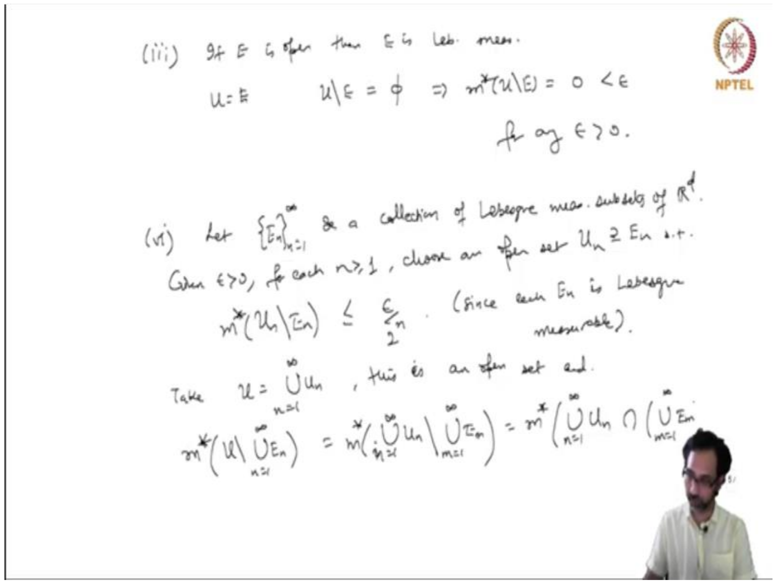

So, let us look at the proof. So, the first one is pretty easy. So, we have already seen that m star phi = 0 this was one of the axioms for the outer measure. And so, we can take the empty set itself as the open set that covers the empty set. So, take u = phi and then if you take E is also equal to phi then m star of u - E is again m star of the empty set, the set difference of m 2 empty sets is again an empty set and this is equal to 0.

So, given any epsilon is less than epsilon for any positive epsilon, which means that the empty set is Lebesgue measurement. Similarly, if m star E = 0, then we can use the outer regularity of m star to find an open set to get. So, given any epsilon greater than 0, we get an open set u containing E such that m star of u is less than or equal to epsilon. So, by monotonicity m star u - E is less than or equal to m star u and this is less than or equal to epsilon, this implies that E is Lesbegue miserable.

Now, let us look at the third claim, which is that every open set if E is open, then E is Lebesgue measurable. So, this is trivially true because you can take u to be E So, that u - E is the empty set and therefore, m start of u - E is 0 and this is less than any epsilon that you can take for any epsilon positive. So, therefore, if E is open then E is Lebesgue measurable. Similarly, rather than taking the fourth one taking the claim for the closed subsets.

Let us look at the sixth one first, and then we will do the fourth and fifth. So, for the sixth one so, let E n be a collection of measurable the Lebesgue miserable subsets of R d then given epsilon greater than 0 for each n greater than or equal to 1 choose an open set u n containing E n such that m star of u n - E n is less than or equal to epsilon over 2 to the power n again there is a 2 to the power n epsilon over 2 to the power n trick here.

So, for each n greater than or equal to 1 using the fact that E n's are Lebesgue measurable. So, this is because since each E n Lebesgue is measurable. So, we can choose such an open set such that the set difference has measured outer measured at most epsilon / 2 to the power n. So, now, take the union of all these open sets u n, n = 1 to infinity this is an open set. So, union of open sets so, it is an open set and if you take the outer measure of u minus the union and equal to 1 to infinity then this is union n = 1 to infinity u n - union m = 1 to infinity E m.

So, this is again can be written as n = 1 to infinity u n intersection with the compliment and m = 1 to infinity E m compliment.

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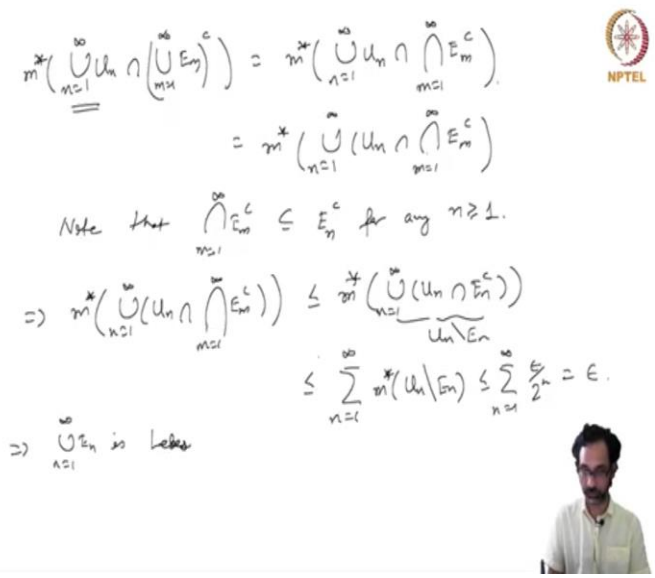

So, let us see what this is. So, we have m star union n = 1 to infinity u n intersection with the compliment of the unions of E ms compliment. So, this is nothing but n = 1 to infinity u n intersection with of m from 1 to infinity E m compliment and so, this can be written as m star union n = 1 to infinity u n intersection with the intersection of all this complements E m compliment.

Now, note that this intersection m = 1 to infinity E m complement is a subset of E n complement for any n greater than or equal to 1. So, this implies that the measure of the Union u n intersection m = 1 to infinity E m complement this is bounded above by the outer measure of u n intersection E n compliment and this is nothing but u n - E n. So, this is less than or equal to the sum n = 1 to infinity m star of u n - E n and each of them is less than or equal to epsilon / 2 to the power n from our choices and this is equal to epsilon.

So, we have shown that we have found an open set u which is the union of all these u n's such that u minus the union of all these E n's has measured at most epsilon. So, the union is union n = 1 to infinity E n is Lebesgue measurable.