Measure Theory

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Lecture – 22

Finite Additivity of Outer measure on Separated Sets, Outer Regularity – Part 2

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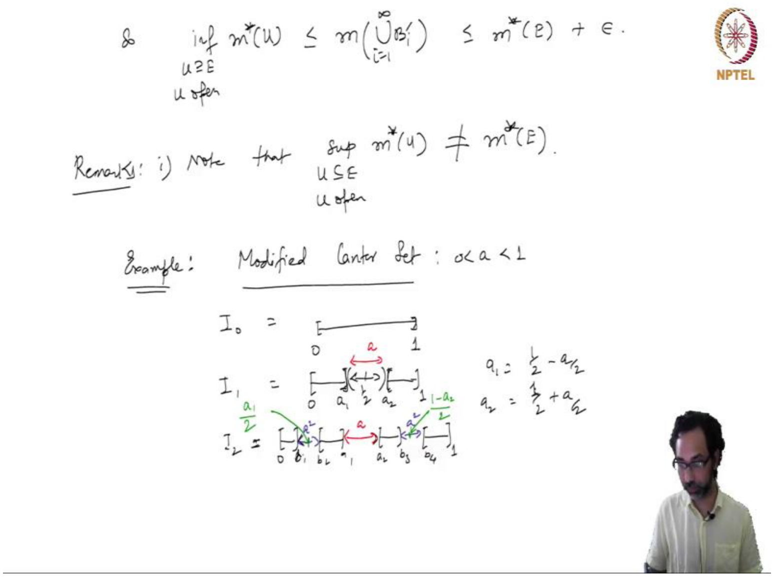

So, the infimum over all open sets u that contain E m star u is less than or equal to the union of the measure of the unions of this B i prime. This is an open set and this is bounded above by m star E + epsilon which show the required result. So, now, let me remark give a couple of remarks. The first is that note that, if we define write the supremum of open sets inside E. So, this is some kind of inner regularity, but with open sets and m star u then this may not be equal to m star E.

This is because since we are considering arbitrary sets E there may not be sufficiently many open sets or then will not be any open sets, which are subsets of E and so, on the left hand side you will get simply a 0, but the right hand side may be non 0. So, I am going to give an example for this. So, this is the so called modified control set. So, the modified control set is defined as follows.

So, u take I 0 is given by the interval 0 to 1, I 1 is given by the interval 0 to some number a 1 and then a 2 to 1 these are all closed intervals, but the middle 1 has length a. So, this is centered at the point half and has length a. So, a 1 is half – a and a 2 is half + a / 2. So, in this way, we have removed an interval a sub interval of length a centered at half. So, now, if you have seen before the canter set in that case, a will be 1 / 3.

And we subsequently remove sub intervals of length 1 / 3 from each of these sub intervals that we get at the end of it, which is which comprises I 1, but here we will not do that. Here in at the second stage, we are going to divide a 1 then. So, let me write b 1, then b 2 to a 1. Then a 2 to b 3, b 3 and then b 4 to 1. So, remember that this length was a, but this length. So, let us use a different color. So, this length was a so this length is a, but these lengths are a square. So of course, here I am taking a, a positive number between 0 and 1.

And at each subsequent step we are removing smaller and smaller sub intervals of length smaller and smaller, because when you take higher powers of a it will become smaller. So, the first one was a at I1. We removed the sub interval of length a centered at half. In the second case, we removed a sub interval of length a square, but centered at a 1 / 2. So, this point here, this point is a 1 / 2 and this point here, this point here is given by 1 – a 2 / 2. So, again, we are taking the center point of each of these sub intervals and then removing sub interval of length a square. So, we continue like this.

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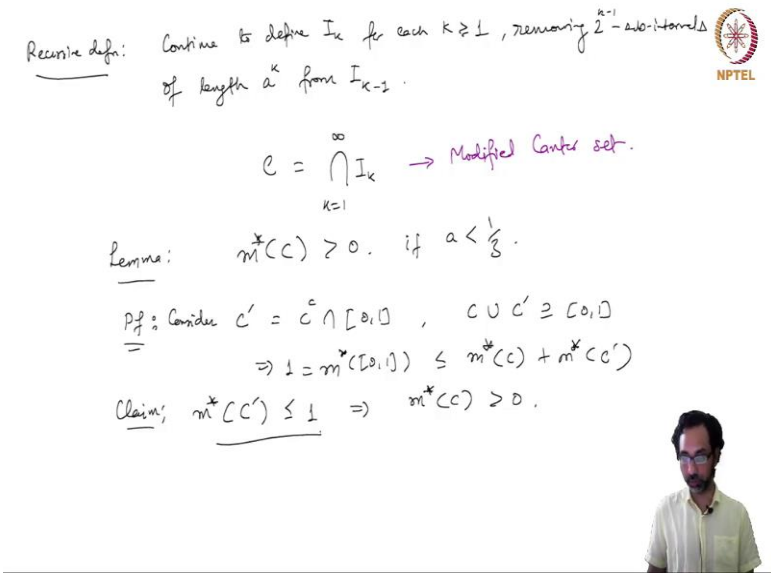

So continue to define I k for each k greater than or equal to 1, removing sub intervals. So, in total at the kth stage, you will have 2 to the power k – 1 sub intervals that are being removed. So, for example, at the first stage, this is 2 to the power 0, so 1 interval and the second stage 2 to the power 1. So, 2 intervals and at the k th stage, u will remove 2 to the power k – 1 sub intervals of length a to the power k from I k – 1.

So we have recursively defined what is I k, this is a recursive definition. And now I am defining c to be the intersection of all these I k's to 1 to infinity, and this is the so called modified canter set and we will show lemma that the modified canter set has strictly positive measure if a is less than 1 / 3. So, let us try to prove this. So, first consider the set c prime. So, consider c prime which is c complement intersection 0, 1.

And we have that c union c prime is contains the set 0, 1. So, therefore by sub-additivity, we will have m star 1 less than or equal to m star c + m star c prime. Now, I am going to show that I claimed that m star of c prime is less than or equal to 1. So, on the left hand side, we have 1 and on the right hand side, we have m star c + m star c prime. So, if m star c prime has value less than or equal to 1, but the sum is greater than or equal to 1, this implies that m star c is strictly positive. So, we have to show that m star c prime is less than or equal to 1.

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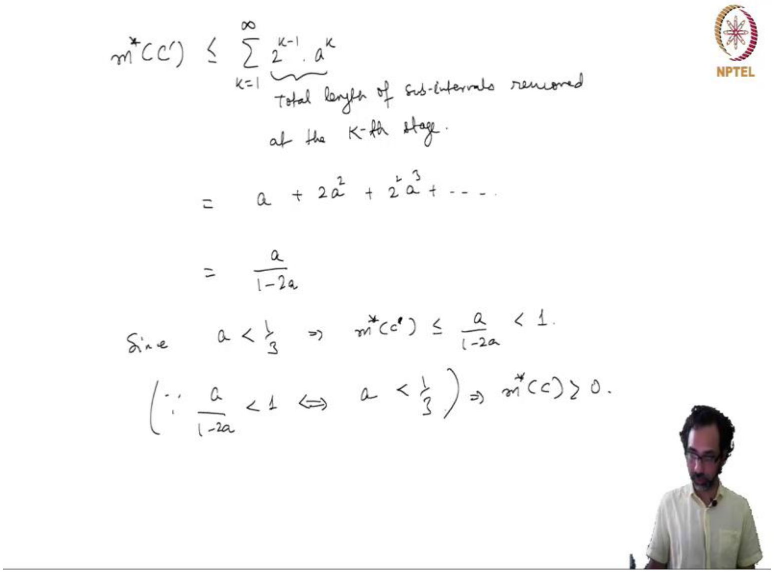

So to show this, note that m star c prime is less than or equal to the sum k = 1 to infinity 2 to the power k – 1 times a to the power k. So, this is the total length of sub intervals removed at the kth stage. So, since c prime is the complement of c restricted to 0, 1. It is covered by the union of all the removed sub intervals and so. The outer measure is bounded above by the sum of the total lengths of sub intervals at the kth stage.

So, we can immediately sum it up so, we will get the sum a, the first term is a then the second term is 2 a, the third term is 2 square, the second term is 2 a square, the third term is 2 square a cube and so on. So, if we compute the sum we will get a over 1 – 2 a. Now, since a is less than assumed to be less than 1 over 3. This implies that m star c prime which is less than or equal to a / 1 – 2 a and this is less than 1 because a / 1 – 2 a less than 1 is equivalent to saying that a is less than 1 / 3.

So, we get m star of c prime is less than equal to 1, which shows that so, we get m star c prime is less than or equal to 1 this shows that m star of the modified canter set is strictly greater than 0.

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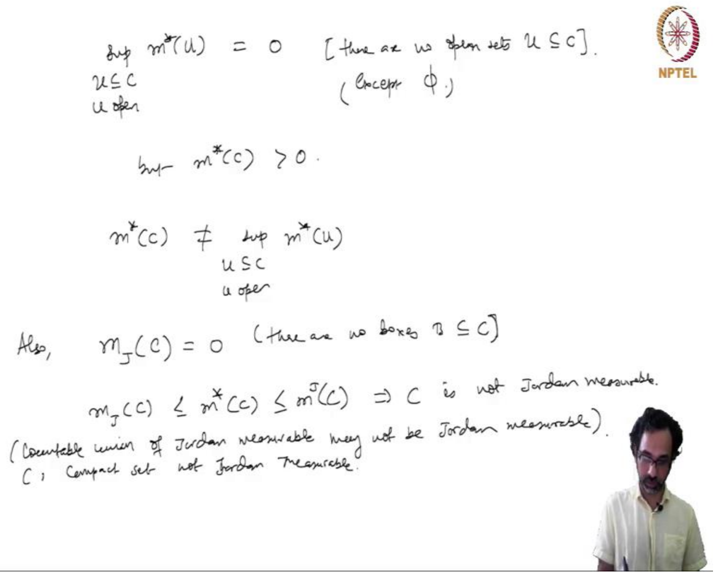

Now, if we write if we take the supremum u inside the c u open m star u then we will get 0 because there are no open sets u sitting inside c. So, except of course the empty set except the empty set. So, we will get 0 here, but m star of c greater than 0. Therefore, the formula does not hold m star c is not equal to supremum of the measures of all open sets inside c of the Lebesgue outer measures m star u.

This example also shows also the Jordan inner measure of this set c is 0 because there are no boxes B sitting inside c. So, the Jordan inner measure is going to be 0. And since the Jordan inner measure is less than or equal to m star c is less than or equal to the outer Jordan measure. This implies that c is not Jordan measurable. So, this is an example of a countable union of Jordan measurable sets.

Because; at each stage the set I k is Jordan measurable because it is a finite union of boxes or intervals. So, at each stage it is Jordan measurable but when u takes the countable union, it fails to be Jordan measurable. So, in fact, this is a even a compact set. And so this is also an example of a compact set not being Jordan measurable. So, compact set, not Jordan measurable. So c is a compact set, which is not Jordan measurable.