Measure Theory
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Lecture – 21
Finite Additivity of Outer measure on Separated Sets, Outer Regularity – Part 1

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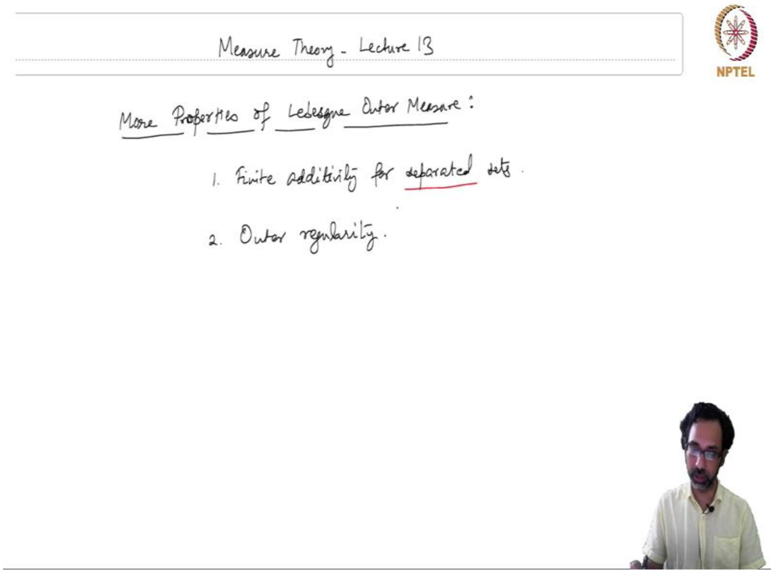

So, let us continue studying some more properties of the Lebesgue outer measure. In this lecture, we will look at 2 more properties, the first one is called finite additivity for separated sets, and the second one is called outer regularity.

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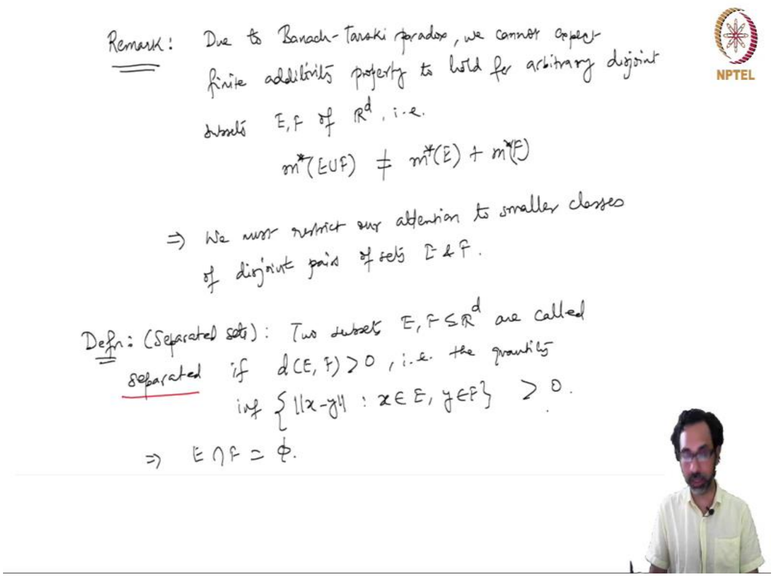

So, let me start with the remark is that, due to the banach tarski paradox that we have seen before due to the banach tarski paradox, we cannot expect finite additivity property to hold for arbitrary subsets E F of R d this is to say that the outer measures of E union. So, arbitrary disjoint subsets of R d which is to say that the outer measure of E union F may not be equal to the outer measure of E plus the outer measure of F.

So, we must this implies that we must restrict our retention to smaller classes of rather than taking arbitrary sets E and F we have to restrict our what kind of sets we allow for considering finite additivity. So, 2 smaller classes of disjoint pairs of sets E and F. So, one such restriction is given by the so called separated sets. So, 2 sets 2 subsets E and F of R d are called separated if the distance between E and F is strictly greater than 0.

This is the quantity given by the infimum of the Euclidean distance between 2 points such that X and Y such that x is in E and y is in f. So, this infimum is strictly greater than 0. So, in this case we call E and F separated. So, of course, if this happens one can also show that this implies that E and F are disjoint because, if E and F have a common point then the Euclidean distance between that point with itself is going to be 0. So, the infimum is will be 0. So, separated implies disjoint.

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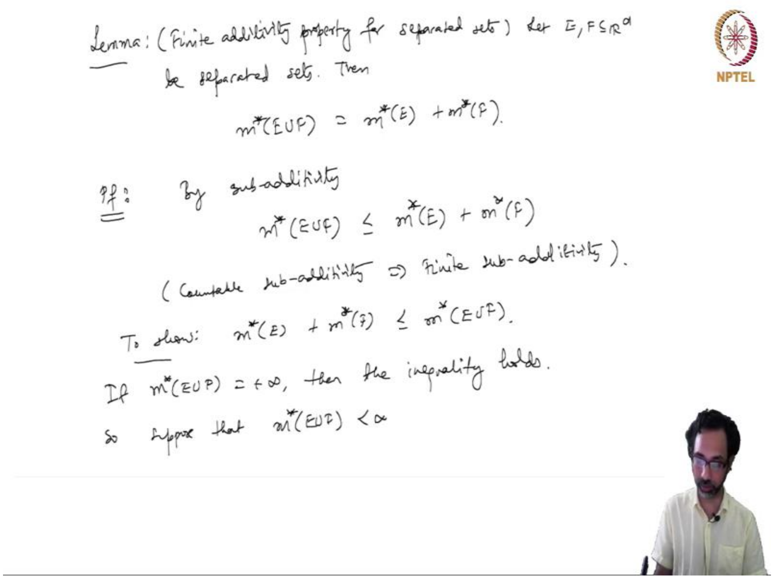

So, the next lemma establishes finite additivity property for separated sets. So, let E and F the subsets of R d which are separated the separated sets. Then, we have the finite additivity property, which is that the outer measure of the union Lebesgue outer measure the union is equal to the sum of the Lebesgue outer measure of E and F. So, let us try to prove this. So, by sub-additivity we have that the measure the outer measure of the union is bounded above by the sum of the Lebesgue outer measure of E and F.

So, we have seen countable sub additivity. So, in this case we only have finite only 2 sets, but we can add infinitely many countable many empty sets and then the measures all for all the rest of the sets will be 0 and we will get this inequality. So, countable sub-additivity implies finite sub-additivity. So, it is enough to show the reverse inequality, which is that m star E + m star F less than or equal to m star E union F. So, again, notice that if m star E union F is plus infinity, then the inequality is stable inequality holds. And so, suppose that m star E union F is finite.

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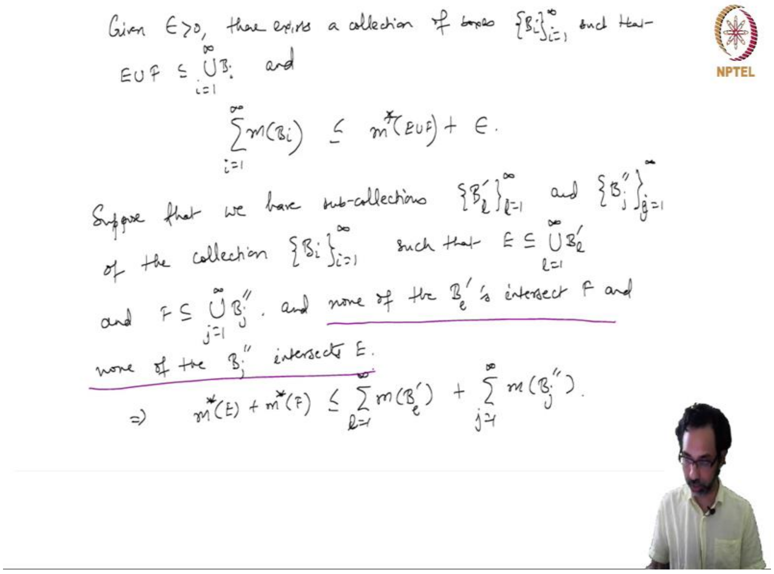

So, now, I am going to again use the infimum definition of the Lebesgue outer measure. So, there exists. So, given epsilon greater than 0, there exists a collection of boxes B i i = 1 to infinity such that E is covered by the union of these B i i = 1 to infinity and the sum i = 1 to infinity of the measures of these B i is bounded above by m star E + epsilon. So, now, suppose that B i suppose that we have sub-collections B i prime i = 1 to infinity and b j double prime.

So, to distinguish the indices, I will write here l for the first one and j for the second one, j = 1 to infinity certainly such that. So, these are sub-collections of the original collection of the collection B i i = 1 to infinity. So, each of these be B l prime and B j double prime are one of these B i s. So, such that E is covered by, I have to cover E union F by this rather than just E, I am going to cover E union F by the whole collection B i and so, now, I am dividing the collection B i into 2 sub-collections, one that covers E.

So, B l prime l = 1 to infinity and F is covered by j = 1 to infinity B j double prime and we also suppose that none of the B l prime intersect F and none of the B j double prime intersects E. So, then these 2 collections are separate there is no overlap between them. So, this implies that m star E + m star F this is bounded above by the sum l = 1 to infinity m of B l prime and then j - 1 to infinity m of B j double prime. But if because of this assumption that we have that none of the B l prime intersect F and none of the B j prime intersect E.

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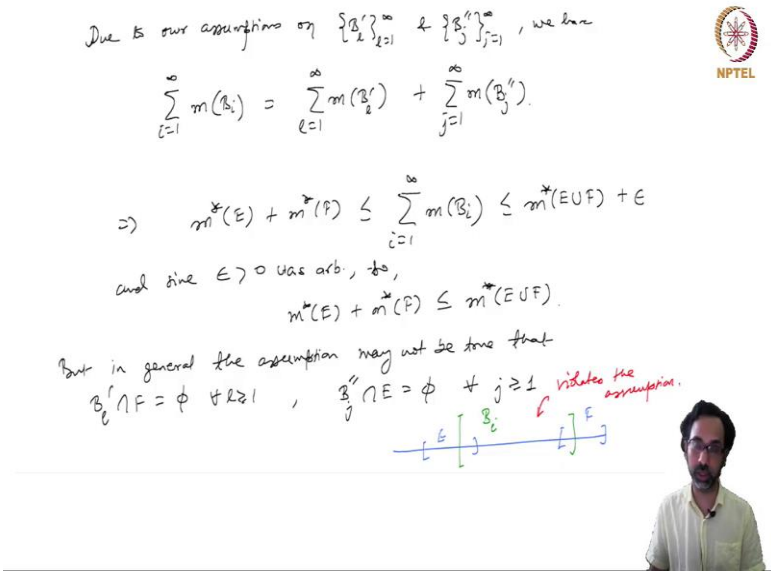

This implies the due to our assumptions on these collections B l prime and B j double prime j = 1 to infinity, we have that the sum i = 1 to infinity of the measures B i = l = 1 to infinity m of B l prime + j = 1 to infinity m of B j double prime. So, this is because there is no overlap between these 2 collections B l prime and B j double prime. So, this implies that the measure of outer measure of E plus the outer measure of F is bounded above by this sum m B i.

But this was chosen such that this is less than or equal to m E union F + epsilon and since, again epsilon is arbitrary was arbitrary. So, we get the required inequality m star E + m star F is less than or + m star E union F. But now, in general the assumption may not be true that none of the so, let me write it in symbols B l prime intersection F is empty for all l greater than or equal to 1 and B j double prime intersection E is empty for all j greater than or equal to 1.

So, this may not be true because, for example, if you have 2 sets E and F. So, there could be one box B j or B i which intersects both and so, our assumption will be invalid, because this set will neither belong to any of the B l primes nor will it belong to any of the B j double primes. So, this will violate our assumption. So, this set violates this assumption. So, what we have to do is to use the fact that these 2 sets are separated and we have to break these big boxes B i into smaller chunks, so, that none of the smaller bits overlap both intersect both E and F.

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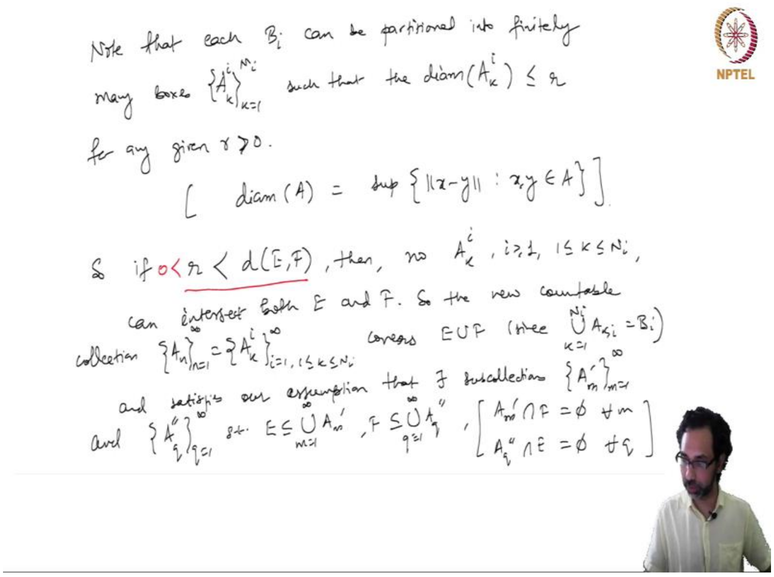

So, note that each B i can be partitioned into finitely many boxes. Now, let me give it a different name A k k = 1 to. So, A i k = 1 to m i, so, for each i, we will have a partitioning of each box B i into finitely many boxes such that the diameter of each A k i is less than or equal to r for any given r greater than 0. So, here the diameter of a set A this is the supremum of the Euclidean distance between any 2 points of A.

So, we can restrict the diameter of the partition boxes, the boxes used in the partitioning of each B i says that the diameter is bounded above by any given positive number r. So, if r is taken to be strictly less than the distance between E and F then no A k i for i greater than or equal to 1 and 1 less than k less than or equal to N i can intersect both E and F. So, that so, the new countable collection of boxes A k i i = 1 to infinity k = 1 to N i.

So, this collection covers E union F because the union when you take the union of these A k i over k this is precisely B i because this is a partitioning of this box B i and this collection satisfies our assumption that there exists sub-collections A. So, rather than taking 2 indices, I can rewrite it re-index it to write it as a collection it A n n = 1 to infinity. So, now, we can break these this collection A n s into 2 sub-collections and A m prime.

And then let us say A q double prime q = 1 to infinity such that E is covered by the first collection and F is covered by the second collection and it also satisfies that A m prime intersection F is empty for all M and A q double prime intersection E is empty for all q. So, this assumption will be satisfied once you subdivide each box B i into smaller boxes with diameter bounded above by the distance d E f. So, note that this is a positive number. So, r can be chosen here greater than 0. So, this takes care of r assumption that we had before and in general case also we can reduce to that case. So, this proves the lemma.

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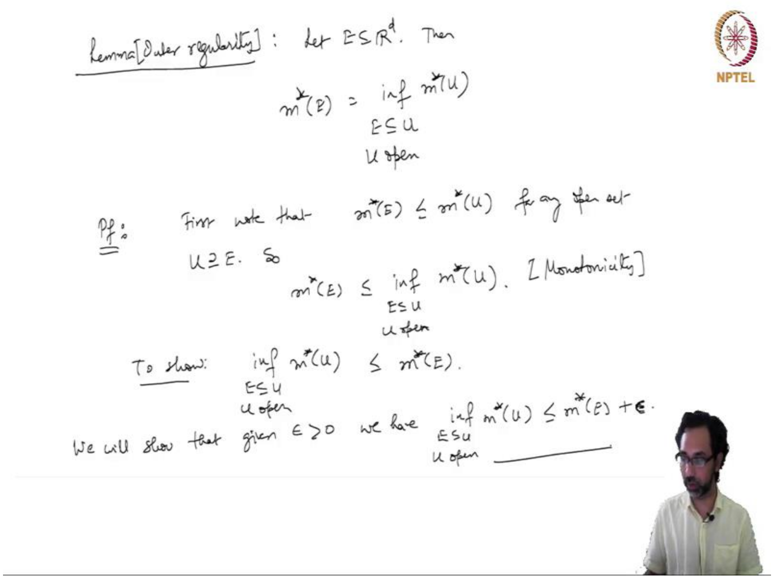

The next lemma is called outer regularity and this says that the outer measure so, let E be a subset of R d then we have a formula for the outer measure given by the infimum of the outer measures of sets u that are super sets of E such that each of these u's are open. So, this is called outer regularity property for the Lebesgue outer measure and this is one of the most important properties and we will see that this property also can be it can also appear in the abstract measures based context.

So, let us try to prove this. So, first note that note that m star E is less than or equal to m star u for any open set u that contains E. So, we can take an infimum on the right hand side and we will get m star E is less than or equal to the infimum of these sets u each of these u's are open and E is contained in each of these open sets u. So, this is obvious from the monotonicity property. Now, so, it is at suffices to show the reverse inequality which is that infimum over open sets of m star u less than or equal to m star E.

So, in particular what we will do is we will show that given epsilon greater than 0, we have infinite infimum of these sets m star u less than or equal to m star E + epsilon. So, again we use the epsilon trick and we will try to show this inequality here.

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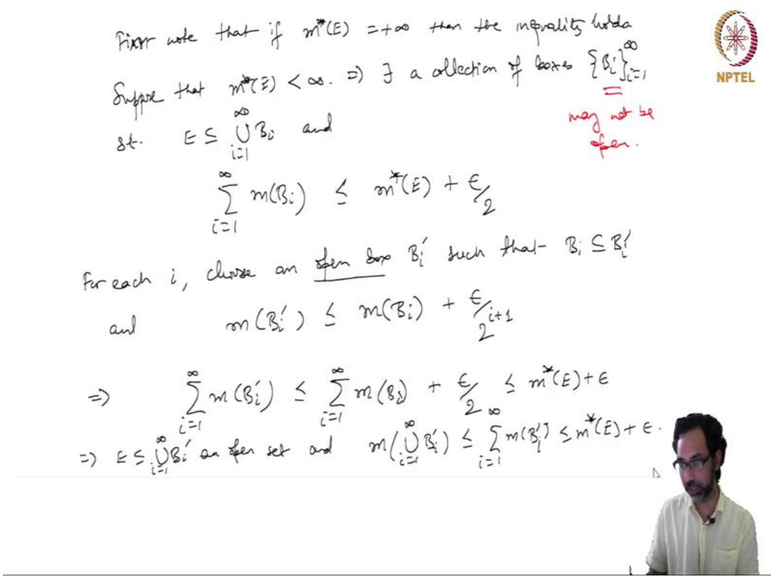

So, to show this first note that if m star E is infinite, then the inequality holds trivially equality holds. So, suppose that m star E is finite. So, in this case this implies that there exists a collection of boxes B i i = 1 to infinity such that E is contained inside this union of the B i's and the sum are equal to 1 to infinity and B i less than or equal to m star e + epsilon. So, let me take epsilon by 2 here. So, now, these B i’s that we have taken these may not be open, but, we can enlarge each of these B i‘s as we have done in the last lecture.

We can enlarge these B i’s such that they become open but still we can have control over the volume of the union. So, for each i choose an open box B i prime such that the measure of B i prime first of all that B i’s contained in B i prime and the measure of B i prime is less than or equal to the measure of B i + epsilon / 2 to the power i + 1. So, this implies that the sum from i = 1 to infinity m of B i prime is less than or equal to the sum i = 1 to infinity m B i and then you will have an extra term of epsilon / 2 but this is less than or equal to m star E + epsilon / 2.

So, these 2 epsilon / 2 s can be written as epsilon. So, we have produced E is covered by these boxes B i prime is equal to 1 to infinity and so, this is an open set, this is an open set and the measures of the union of i = 1 to infinity B i prime less than or equal to this sum i = 1 to infinity and B i prime which is bounded above by m star E + epsilon. So, we have produced an open set such that it is bounded above by m star E + epsilon. Therefore, when we take the infimum over all open sets, then it is going to be less than this quantity.