Measure Theory
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Lecture - 20
Comparing Inner Jordan Measure, Lebesgue Outer Measure and Jordan Outer Measure

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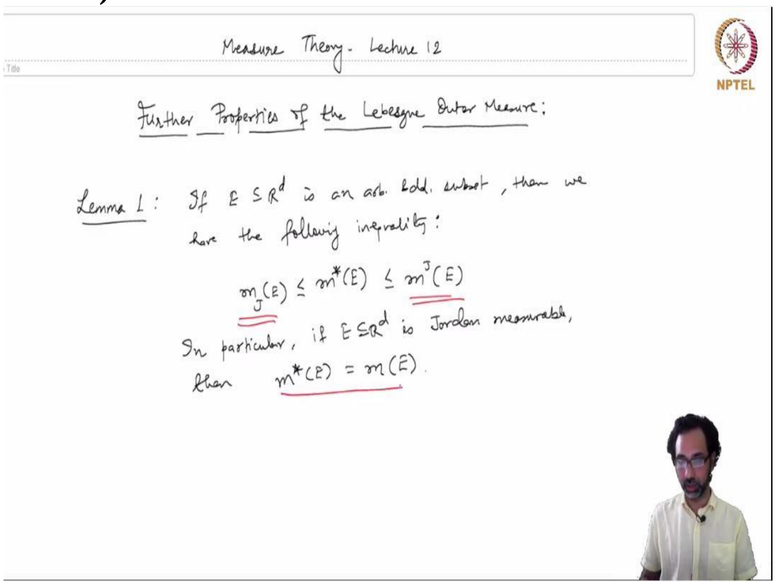

We continue our study of the lebesgue outer measure and today we are going to learn some more properties of the lebesgue outer measure. So, the first one I will put this as a lemma 1 this says that if E is a subset of R d which is bounded is an arbitrary bounded subset then we have the following inequality which compares the lebesgue outer measure m star E with that of the Jordan inner and outer measures. So, we have already seen that the lebesgue outer measure is bounded above by the Jordan outer measure.

But it is also bounded below by the Jordan inner measure. So, in particular if E is Jordan measurable then we have the equality of the lebesgue outer measure with the Jordan measure of E. So, this also holds for elementary sets because elementary sets are Jordan measurable and so, with this inequality we can deduce because if E is Jordan measurable, then these 2 things the inner and outer Jordan measures are equal and so, we have an equality with the lebesgue outer measure of the Jordan measure.

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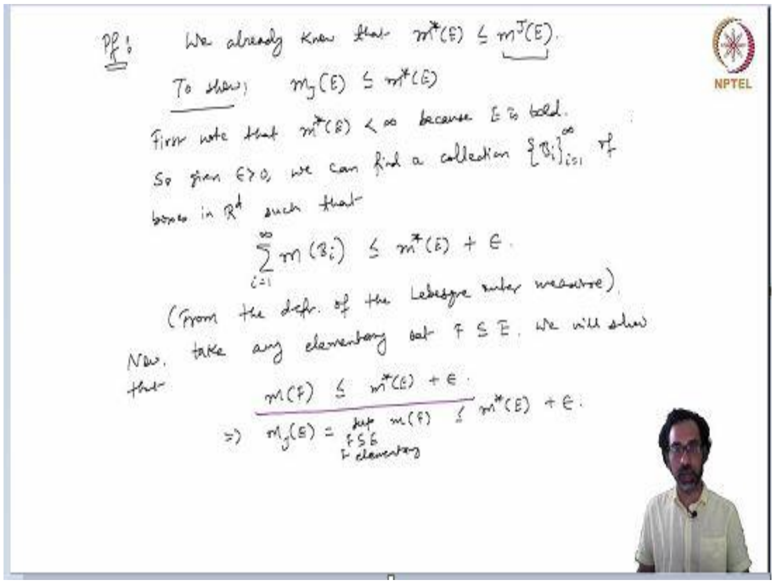

So, let us try to show this result. So, we already know that m star E is bounded above by the lebesgue by the Jordan outer measure. So, we have to show that the Jordan inner measure is bounded above by the lebesgue outer measure first note that m star of E is finite because is bounded. So, if E is bounded then the Jordan outer measure is finite we have already seen this. So, in particular the lebesgue outer measure is also finite. So, given epsilon greater than 0, we can find collection B i = 1 to infinity of boxes in R d.

Says that we have m star of this sum i = 1 to infinity m star of B i rather m of B i is less than or equal to m star E + epsilon. So, this is from the definition of the lebesgue outer measure from the definition of the lebesgue outer measure. Now, take any elementary set F inside E. So, we will show that the elementary measure of F which sits inside E is bounded above by m star E + epsilon. So, one can then take the supremum on the left hand side and we get.

So, the inner Jordan measure elementary m F is then bounded above by m star E + epsilon and because epsilon was arbitrary we would be done. So, we have to show this inequality that m F is bounded above by n star E + epsilon.

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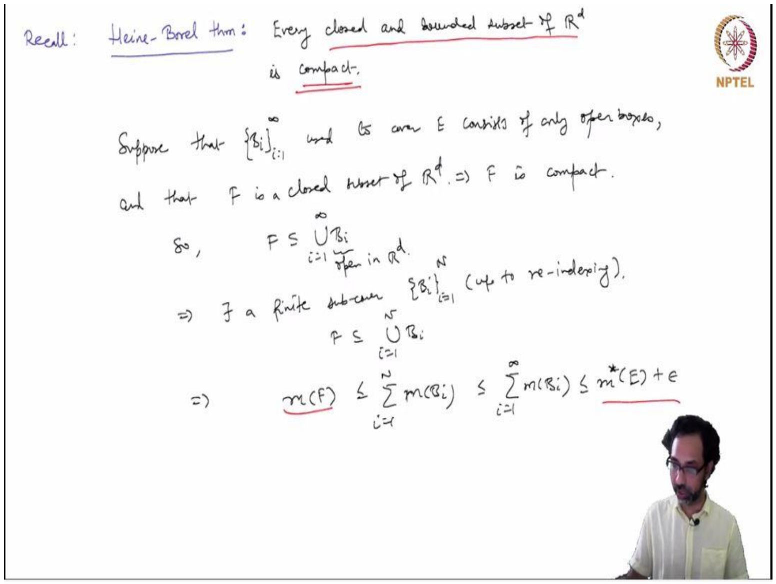

So, to prove this we need the following fact from topology which is called the Heine-Borel theorem. So, we recall it here it says that every closed and bounded subset of R d is compact, compact meaning that every finite every open cover has a finite sub covering by definition. So, this result is a deep converse for the fact that every compact set in R d is must be closed and bounded. So, this is a converse saying that every closed and bounded subset is also compact. So, we would need this fact.

So, suppose, so, let us see first how we will apply this. So, suppose that this covering that we used B i, i = 1 to infinity used to cover E is made above consists of only open boxes. So, to apply the Heine-Borel of theorem, which is to apply compactness, we would need the R covering to be open. So, that is why we are supposing that each of these box B i is open suppose also and that F is closed is not is a closed subset of R d. So, because E was bounded, if F is closed F is also bounded, this implies that F is compact.

And so, F is covered by the union of these boxes B i and each box is open in R d. So, because F is compact there exists a finite sub cover of these open covers B i i = 1 to N. So, up to reordering up to reindexing. So, if needed we can reindex this collection B i and we can take all the needed boxes open boxes B i for this sub cover finite sub cover and arrange it from 1 to N. So, we can write that this finite sub cover is the collection B i from i = 1 to N and so, the measures of F.

So, of course, the finite sub cover means that F is contained in this finite union of boxes of B i i = 1 to N. So, this implies that the measure of F is bounded above by the finite sum from i for i = 1 to N and B i and this is bounded above by the infinite sum because we have only non negative values. So, we can extend this sum to the infinite sum equal to 1 to infinity m B i and this is then bounded above by m star E + epsilon.

So, if F is closed and if these B i that are used to cover E consists only of open boxes then, we can immediately find our required inequality which is m F is less than or equal to m star E + epsilon. But, because this is not always the case, so, we have to do some more carefully.

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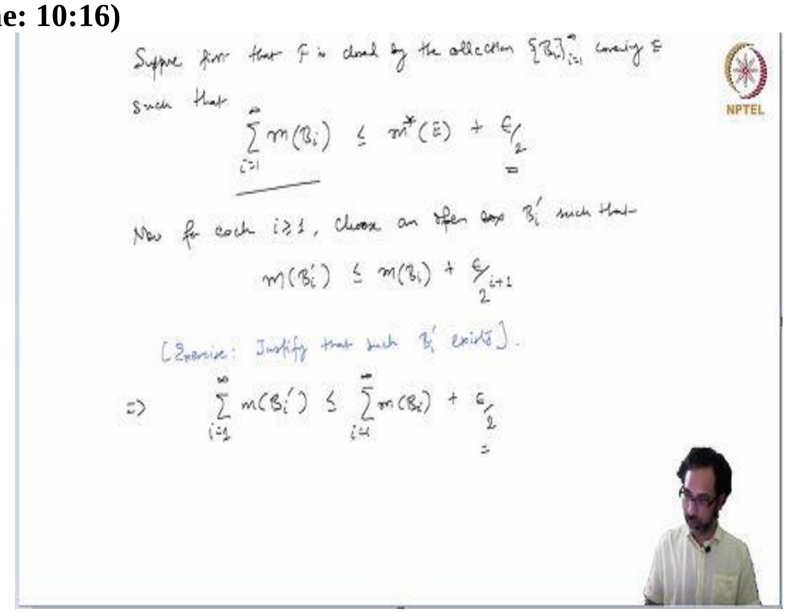

We have to investigate a little bit more on how to reduce it to this case. So, suppose first that F is closed, but the collection B i, i = 1 to infinity covering E such that this sum i = 1 to infinity m B i is less than or equal to m star E + epsilon / 2. So, we can choose such a collection of boxes B i countable collection covering E says that the sum the infinite sum is bounded above by m star E + epsilon / 2.

Now, for each i greater than or equal to 1 choose an open box B i prime such that m of B i prime is less than or equal to m of B i + epsilon / 2 to the power i. So, again we are going to use the epsilon / 2 to the power k trick, because we want to sum it up in the end, so, there is an epsilon / 2 to the power i and this is an interesting exercise in its own exercise, justify that such a B i prime exists. So, here, it is a matter of enlarging the box B i such that the volume of the enlarged open box B i prime is very close to the original volume of the original box B i.

Up to some epsilon. So, in fact, one can choose any arbitrary epsilon and one can still find an open box B i prime such that m B i prime is less than or equal to m B i + epsilon, that is all we need to prove. So, it is an exercise I which I lead to you to justify that such B i prime exists such open boxes exist. So, given this implies that sum equals from i to i from 1 to infinity m B i prime is less than or equal to some i = 1 to infinity m B i plus we will have an epsilon extra epsilon here. So, let me write here 2 to the power i + 1, so, we will get epsilon / 2. So, this epsilon / 2 and epsilon / 2 above are going to give us epsilon.

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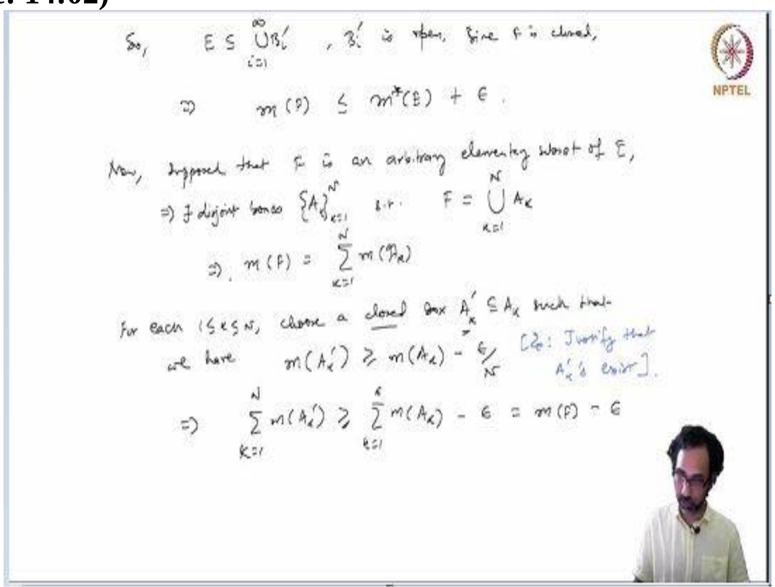

So, now we have a covering of E by these boxes B i prime and each B i prime is open. So, if since F is closed, we are back to the previous result that we showed. So, this implies that m F is less than or equal to m star E + epsilon. So, this shows that when we assume F to be closed, but B i is can be an arbitrary collection of boxes, then we can enlarge each of these box, keeping the volume keeping control over the volume of these boxes.

Said that in the end we have a collection of open boxes and we still get the result that we want. Now, we still have to prove the case when F is not closed. Now, suppose that F is an arbitrary elementary set elementary subset of E. So, this implies that there exists boxes let me write this as A case, k = 1 to capital N says that F is equal to the union of this A case, k = 1 to N. So, we can also assume that these boxes are disjoint. So, our m F is equal to the sum of these measures m A case k = 1 to N.

So, we have an arbitrary elementary set F, which is then expressed as a finite union of disjoint boxes A k. Now, we are going to replace these A case by closed boxes, such that the measures do not change too much. So, for each k greater than equal to 1, so, k between 1 and N choose a closed box A k prime inside A k such that we have the measure of A k prime is greater than or equal to measure of A k - epsilon by N so, for each k, we are choosing a closed box sub box of this box A k.

Which we denote by A k prime such that the volume of this smaller box is between is greater than or equal to m A k - epsilon / n. So, again, it is an exercise to justify that such a case A k primes exist. So, in the previous case, we used an enlargement of an arbitrary box to an open box and in this case, we had going to shrink the boxes a little bit so, that the becomes a closed box, but still we have control over the volumes. So, now, this implies that the sum k = 1 to N m A k prime is greater than or equal to the sum k = 1 to N m A k minus we will get epsilon because there are n terms so we will just get epsilon. So, the right hand side is nothing but m F - epsilon.

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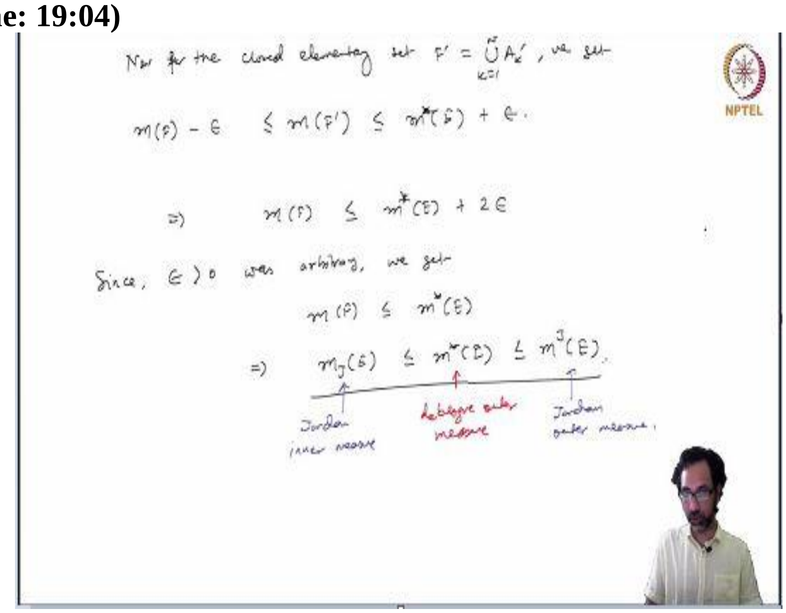

Now, for the closed elementary set F prime which is the union of these A k primes k = 1 to N, we get m of F prime is less than or equal to m star E + epsilon because we have already shown that when we have a closed elementary set, then this inequality is satisfied. So, for the closed elementary set F prime we can write this inequality and m of F prime is then bounded below by m F - epsilon. So, this is implies that m F is bounded above by m star E + 2 epsilon. But since epsilon is arbitrary we get m F is less than or equal to m star E.

And finally, this implies that the inner Jordan measure is bounded above by m star E which is bounded above by the outer Jordan. So, finally, we have proved what we wanted, which is that the lebesgue outer measure is sandwiched. So, this is the lebesgue outer measure it is sandwiched between the Jordan inner measure and the Jordan outer measure for bounded subsets of R d. So, in particular when each Jordan measurable then the inner Jordan measure and the outer Jordan measure go inside and they will give you the same result as the lebesgue outer measure.

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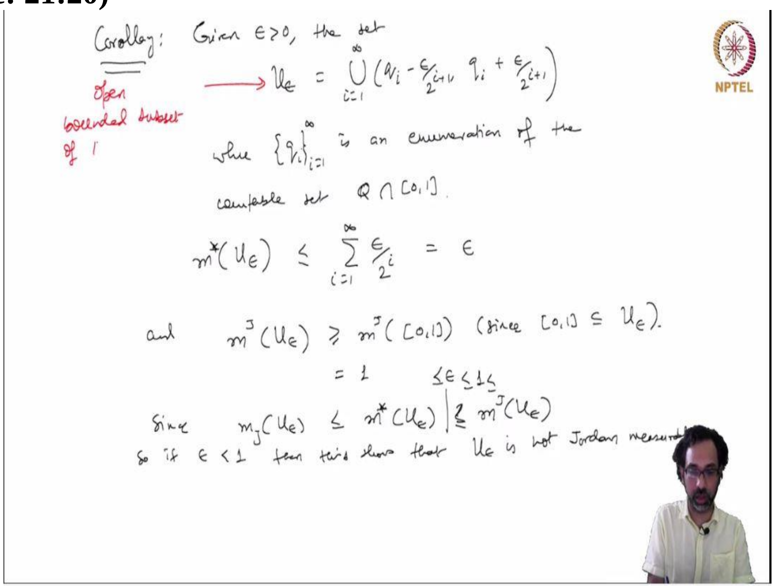

So, one corollary is that the set given epsilon greater than 0 the set U epsilon is equal to the union i = 1 to infinity q i - epsilon / 2 to the power i + 1 q i + epsilon / 2 to the power i + 1. So, where q i i = 1 to infinity is an enumeration of the countable, set of rational numbers inside 0 1. So, we have seen this argument, we have seen this example before where we I claim that this is not Jordan measurable. And we will prove this here. So, now we can compute the lebesgue outer measure of U epsilon.

And this is less than or equal to the sum of epsilon / 2 to the power i = 1 to infinity. And this is just absolutely. And the Jordan outer measure of U epsilon is equal to the Jordan outer measure. Well, it is greater than or equal to the Jordan outer measure of 0 1. Since 0 1 is a subset of U epsilon because, the rationals are dense in 0 1 the set 0 1 the interval 0 1 is a subset of U epsilon. So, we have by monotonicity that the outer Jordan measure of U epsilon is greater than or equal to 1 while the living outer measure is less than or equal to epsilon.

So, since the inner Jordan measure is less than equal to the lebesgue outer measure is less than or equal to the Jordan outer measure this is bounded above by epsilon while this is bounded above by 1. So, if you choose epsilon small enough, so, if epsilon is strictly less than 1, then we shows that U epsilon is not Jordan measurable. So, this is an example of an open bounded set. So, this is an open bounded subset of R which is not Jordan measurable.

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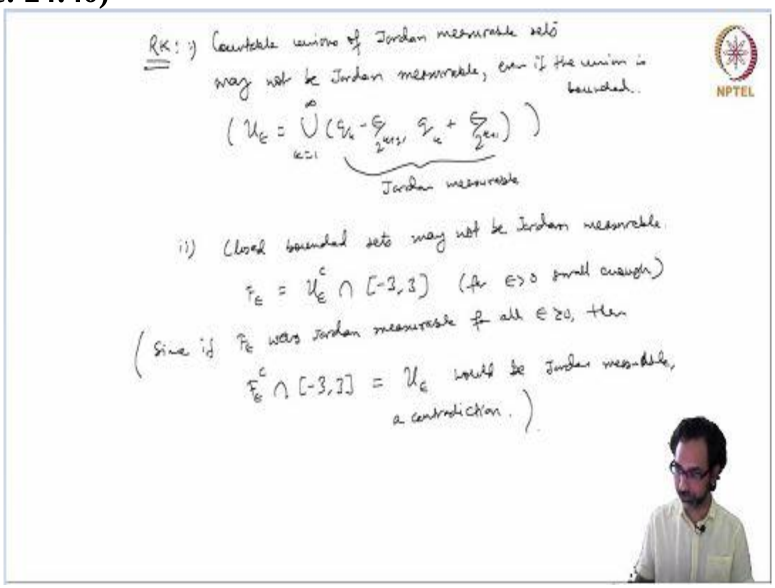

This also shows that the following fact that so, I put it as a remark that countable unions of Jordan measurable sets may not be Jordan measurable. So, we mentioned this before and with the help of the lebesgue outer measure we have shown that countable unions of Jordan measurable sets may not be Jordan measurable in this case, our example was this open set U epsilon which was a countable union of intervals k = 1 to infinity q k - epsilon / 2 to the power k + 1 q k + epsilon / 2 to the power k + 1.

So, each of these is Jordan measurable this is just an interval so, it is a Jordan measurable set, but the union is not Jordan measurable, even if the union is bounded. So, in this case the union is of course bounded even if the union is bounded. So, one can have unbounded unions, but their Jordan measurability is not defined yet, but, even if the union is bounded, we have shown that it may not be Jordan measurable. So, this is the first one second one is that closed bounded sets may not be Jordan measurable.

So, we can take for example, F epsilon, which is the compliment of U epsilon U epsilon compliment intersecting with some big interval let us say - 3 to 3 and this set is not Jordan measurable because if it was Jordan measurable, so, it is not very difficult to see that F epsilon is not Jordan measurable for epsilon small enough of course, since, if F epsilon was Jordan measureable for all epsilon, then, you can take the compliment of F epsilon, which would then be Jordan measurable if you intersect it with say - 3 3.

Now, this will be a Jordan measurable set, but one can show that this is nothing but U epsilon. So, this is a contradiction then would be the Jordan measurable this is a contradiction. So, if epsilon is not Jordan measurable, so, we have seen that open bounded sets as well as closed bounded sets may not be Jordan measurable.

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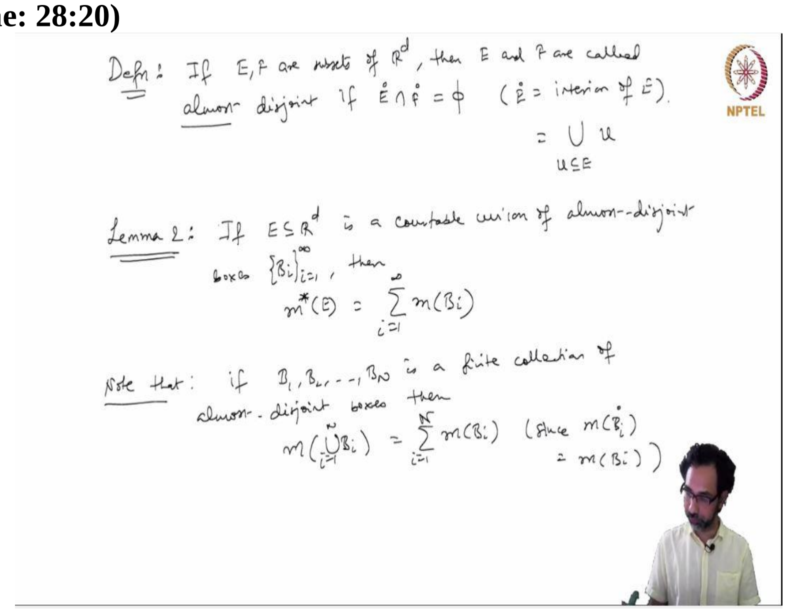

Now, we come to our next property and to state this property we made the following definition if E and F are subsets of R d arbitrary subsets, then E and F are called almost disjoint if the interior of E is disjoint with the interior of F so, here E interior E I am using this notation is the interior of E and similarly for F. So, almost disjoint means that once you take the interior remember that the interior is the union of all open sets U inside E. So, once you take the interior and of E and once you take the interior F then you get an empty intersection.

So, the following lemma uses the concept of almost disjoint sets. So, if E is a subset of R d is a collection is a countable union of almost disjoint boxes then the lebesgue outer measure is given by the sum i = 1 to infinity. So, let me write B i to be this collection, i = 1 to infinity to be this collection of almost disjoint boxes then the lebesgue outer measure is the sum i = 1 to infinity m B i. So, here note that if B 1 B 2 and B n is a finite collection of almost disjoint boxes.

Then the measure of the union of this B i is = 1 to N is equal to the sum of these B i because, since measure of the interior of B i is the same as the measure of B i. So, this lemma extends this fact to countable union of almost disjoint boxes.

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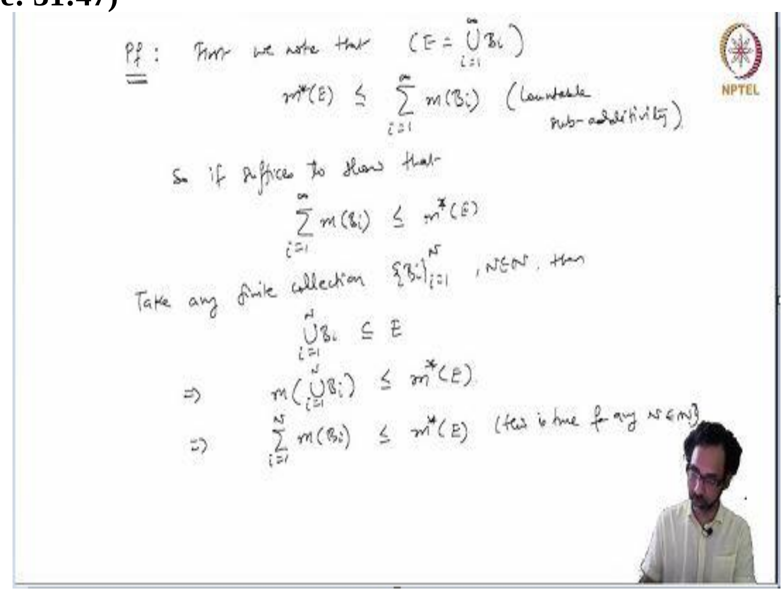

So, let us try to prove this proof is not very difficult. So, first we note that the outer measure of E. So, E here is remember the countable union of these boxes which are almost disjoint. So, by countable sub additivity we get that the lebesgue outer measure of E is bounded above by the sum of the measures of this B i i = 1 to infinity this is from countable sub additivity so, it suffices to show that reverse inequality that the sum is bounded above by m star E. So, this is pretty easy.

So, for example, if you can take any finite collection take B i i = 1 to capital N for some n in the natural numbers, then the union i = 1 to N of B i sits inside this set E. So, by monotonicity we have that the measure of the set is less than or equal to m star E but this is the same as this sum i

= 1 to N measure of B i is bounded above by m star of E now, you can so, this is true for all N this is true for any N E N.

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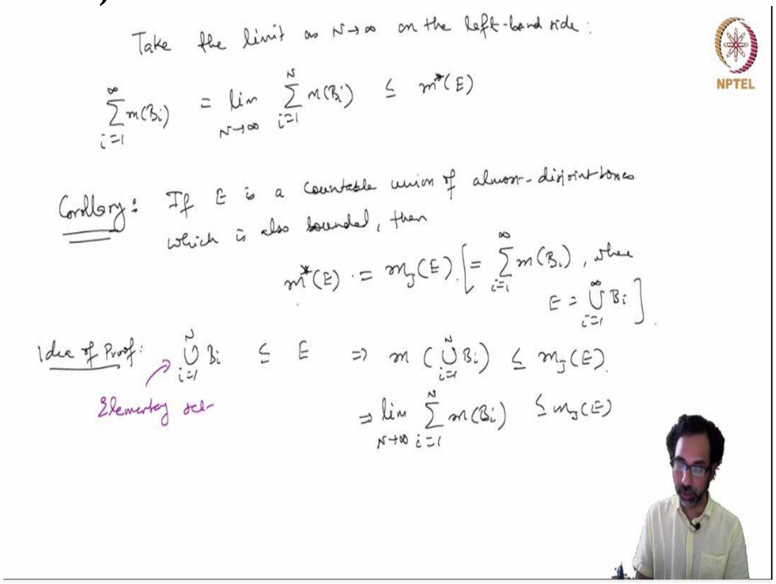

So, we can take the limit as N goes to infinity on the left side left hand side. So, we get the limit as N goes to infinity i = 1 to N and B i is bounded above by m star E. But this is nothing but this infinite sum by definition of in an infinite sum. We get the required bound. So, as a corollary of this result it can be shown that if E is countable union of almost disjoint boxes which is also bounded. Which is also bounded then the lebesgue outer measure is equal to the inner Jordan measure of E.

Because both of them are equal to the sum or equal to 1 to infinity m of B i, where E is this union of B i. So, of course, we have shown that the lebesgue outer measure is equal to this sum to show that the inner Jordan measure is also equal to this sum, we can again apply that this finite unions i = 1 to N. So, this is now an elementary set this is an elementary set and so, this is contained in E. And so, the lebesgue the Jordan measure of this set B i is less than or equal to the inner Jordan measure of E and this is nothing but again, i = 1 to N m B i.

Because this is an elementary set so, we can remove the Jordan, the inner Jordan measure, it is just the elementary measure and so, this is bounded above by E and then again you can take the limit as n goes to infinity. So, this is the idea of the proof which is the same essentially the same as the lemma before. So, we see that the inner Jordan measure for a countable union of almost disjoint boxes, which is bounded is then equal to the lebesgue outer measure.