**Measure Theory**

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**Lecture – 2**
**Infinite Sets and the Banach-Tarski Paradox - Part 1**

So, in the last lecture, we saw the notion of cardinality of a finite set and we saw the definition of the cardinality is not ambiguous, meaning it is well defined. We also showed that the cardinality follows this finite additivity property by which if you take two disjoint non-empty sets, then the cardinality of the union of the two sets will be equal to the sum of the cardinalities of the individual sets.

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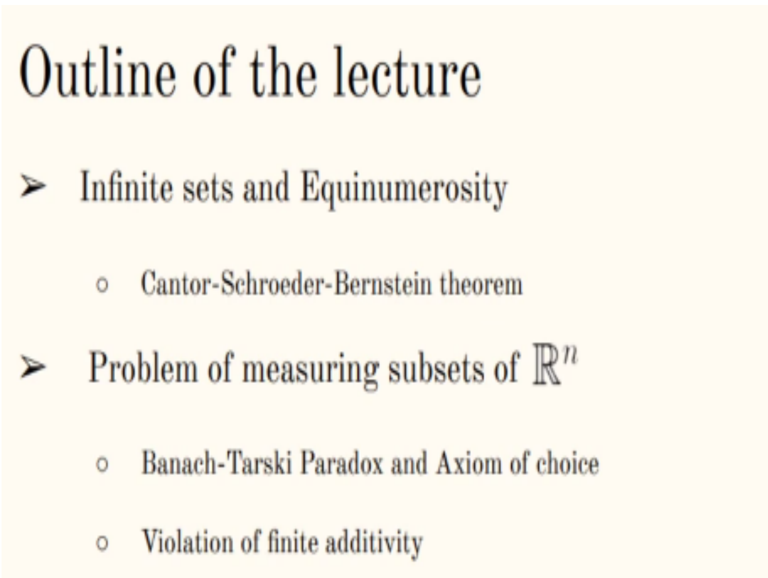


Today, we will see the notion of an infinite set and even though we cannot have a reasonable theory of cardinality of infinite sets, but we can still have a notion of measuring the relative size of two infinite sets via injective or bijective or surjective functions and which is called equinumerosity. So, one of the main theorems in this topic is the Cantor-Schroeder-Bernstein theorem, which gives you a condition under which two arbitrary infinite sets are equinumerous meaning that there exists a bijective correspondence between the two.

Now, when it comes to measuring subsets of the Euclidean space $\mathbb{R}^n$, we will show that the notion of cardinality and equinumerosity will not help us because it will violate our fundamental geometric intuitions of length, area or volume and we will show that given some reasonable geometric rules like finite additivity and invariance under translations, reflections and rotations, if we allow our new assignment of numerical values to arbitrary subsets of $\mathbb{R}^n$ to follow these rules, then we arrive at a contradiction and this is called the Banach-Tarski paradox, which says that if you can start with a solid unit ball in $\mathbb{R}^3$ and you can divide it into finitely many pieces which can then be reassembled to form two disjoint copies of the original ball and so our notion of finite additivity will be violated in this case.

The Banach-Tarski paradox uses in a fundamental way, the so called Axiom of choice which is an axiom in set theory and we will see that with this Axiom of choice, the Banach-Tarski paradox holds and the violation of the finite additivity for subsets of $\mathbb{R}^n$ forces us to categorize subsets of $\mathbb{R}^n$ into two groups, one will be called measurable subsets and the other will be the non-measurable subsets.

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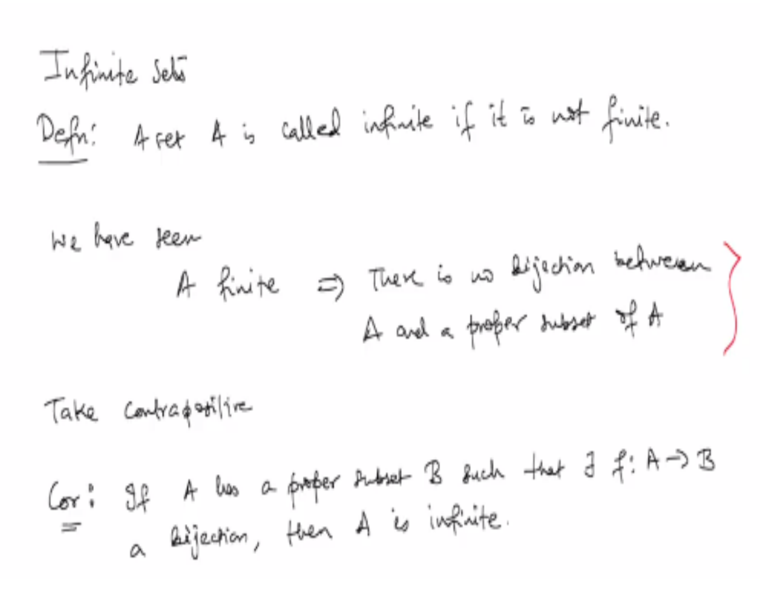


Now that we have seen the basic properties of cardinality of finite sets, let us turn our attention to infinite sets. So, by definition a set $A$ is called infinite if it is not finite. So, since we have seen already that a set $A$ is finite implies that there is no bijection between $A$ and a proper subset of $A$. So, if you take the contrapositive of this statement, we get this corollary that if $A$ has a proper subset $B$ such that there exists a bijection between $A$ and $B$, then $A$ is an infinite set.

So, this is just by taking the contrapositive of this statement about finite sets. So now, if we talk about cardinality of infinite sets, then this notion is not well defined.

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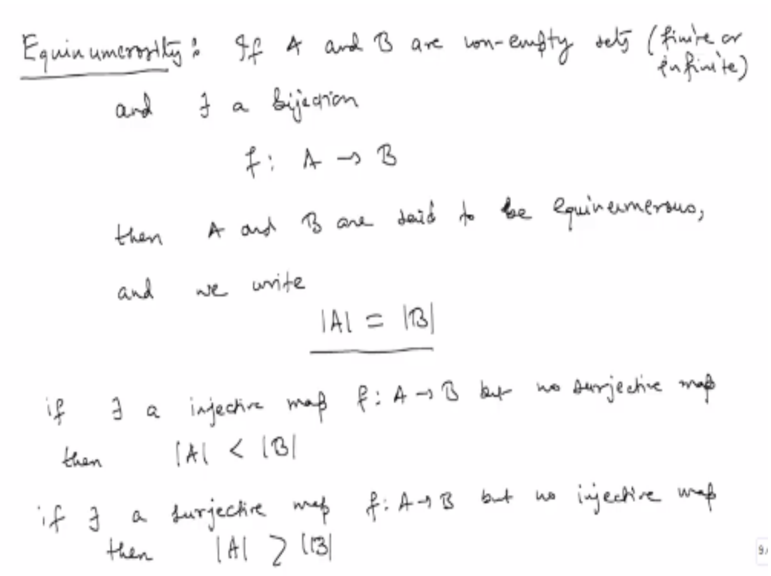


So, we still want some method or some way to compare the sizes of two possibly infinite sets and we again make use of the existence or non-existence of bijective maps between two sets, but now we can also consider injective maps and surjective maps. So, this leads us to the concept of equinumerosity. So, if $A$ and $B$ are non-empty sets and these non-empty sets can possibly be finite or infinite, and there exists a bijection between $A$ and $B$, then $A$ and $B$ are said to be equinumerous and we write this following notation $|A| = |B|$, modulus of $A$ is modulus of $B$. So, of course, we can call it the cardinality of $A$, but this cardinality does not have any defined finite numerical value, but still we can write this as a sort of equation between infinite objects. So this is a further notation.

So if there exists an injective map $f: A \to B$, but no surjective map, then we write that $|A| < |B|$, and if there exists a surjective map $f: A \to B$ but no injective map, then the $|A| > |B|$.

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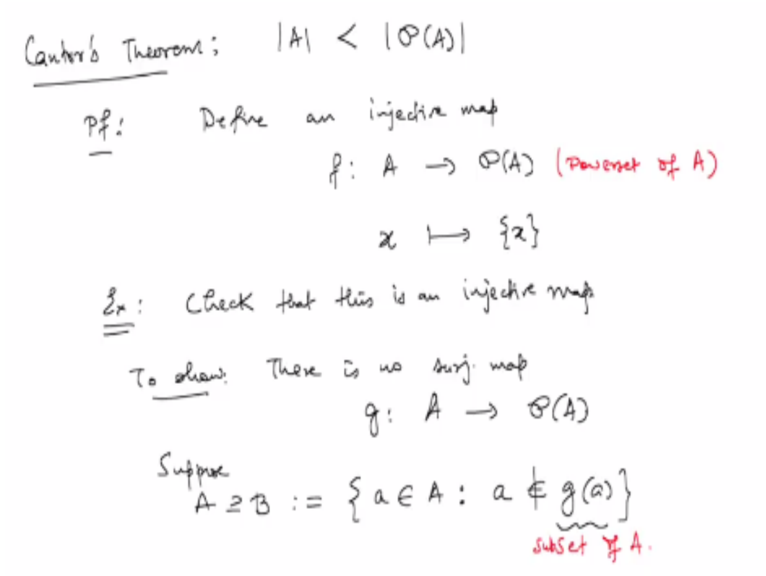


So, the question naturally arises what are the conditions under which we can compare two possibly infinite sets and one such condition is given by Cantor's theorem which says that the cardinality of $A$ is strictly less than the cardinality of the power set of $A$, that is $|A| < |\wp(A)|$. So, here one has to show that there exists an injective map between $A$ and the power set of $A$, but there is no surjection.

So, it is easy to produce an injective map. So, we can define an injective map as follows.: $f : A \to \wp(A)$,
$\wp(A)$ is the power set of A, that is set of all subsets of $A$. So, this map takes an element $x$ in $A$ to the set $\{x\}$ and one can easily check that this is an injective map. So, I will leave it as an exercise for you to check that this is an injective map. Now, the second part is to show that there is no surjective map $g : A \to \wp(A)$. So, we again try to prove this by contradiction. So, suppose we take the following subset $B$ of $A$. So,

$$A \supseteq B := \{a \in A : a \notin g(a)\}$$

Now, note here that $g(a)$ is a set and it is a subset of $A$ and so we are asking that, this collection B should be all the elements of $A$ such that $a$ does not belong to this particular subset $g(a)$ of $A$.

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Claim: $B$ does not belong to the range of $g$.

Suppose to the contrary that $B = g(a_0)$ for some $a_0 \in A$.

Q: Does $a_0 \in B$ or $a_0 \notin B$?

$$a_0 \in B \iff a_0 \in A \setminus g(a_0)$$

$$\iff a_0 \in A \setminus B \rightsquigarrow \text{a contradiction.}$$

$$a_0 \notin B \rightsquigarrow \text{a contradiction}$$

Thm. [Cantor-Bernstein-Schroeder] If $A$ and $B$ are non-empty sets and $f: A \to B$ is an injective map and $g: B \to A$ is also an injective map, then $|A| = |B| \iff \exists$ a bijection $h: A \to B$.

Now, we claim that $B$ does not belong to the range of this map $g$. Now to show this, we ask the following question that suppose first that to the contrary, that $B = g(a_\circ)$ for some $a_\circ$ in A, which means that $B$ belongs to the range of $g$. And now we ask the question does $a_\circ$ belong to $B$ or $a_\circ$ does not belong to $B$ ? So, this question will lead us to an abstract conclusion which is as follows.

So, suppose that $a_\circ \in B$, then this is equivalent to saying that $a_\circ \in A\backslash g(a_\circ)$, $g(a_\circ)$ is a subset of $A$ and you can take $A\backslash g(a_\circ)$, but this means that $a_\circ \in A\backslash B$, and this is a contradiction. So, similarly, one can ask if $a_\circ \notin B$ and this also leads to a contradiction. So, in both cases we had led to a contradiction. This is the end of the proof.

Now, another famous theorem which deals with equinumerosity is as follows and this is a theorem which is named after Cantor, Bernstein and Schroeder. It says that if $A$ and $B$ are non-empty sets and $f : A \to B$ is an injective map and $g : B \to A$ is also an injective map, then $A$ and $B$ are equinumerous. So, the cardinality of $A$ is equal to the cardinality of $B$, equivalently there exists a bijection $h$ between $A$ and $B$.

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Pf: See Folland's book (introducti

So, I will not prove it here, but rather I refer you for the proof see Folland's book and this is in the introduction chapter on set theory. So, a concise proof can be found there.