Measure Theory

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Lecture - 19
Outer Measure - Motivation and Axioms of Outer Measure

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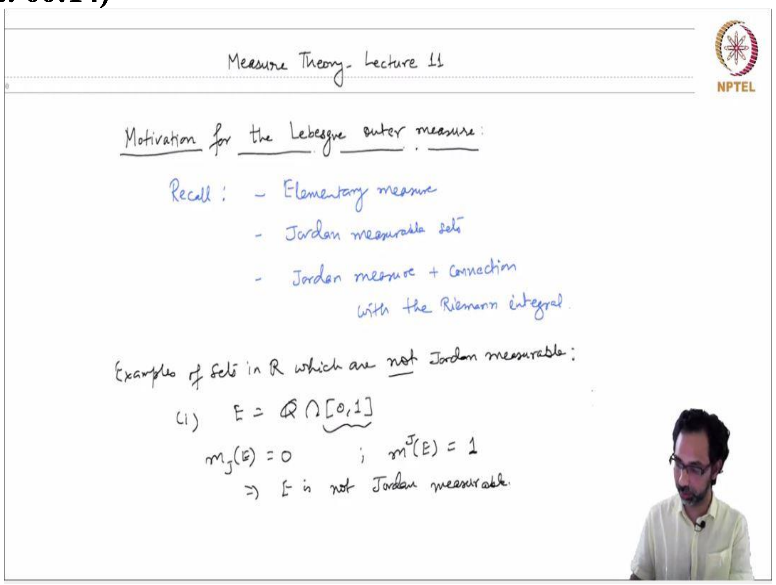

Let us begin with some motivation for the lebesgue outer measure. So let us recall that until now, we have seen the concept of elementary measure, the Jordan measurable sets and the Jordan measure, Jordan measure and its connection with the Riemann integral. So, to motivate the lebesgue outer measure, let us see some examples of sets which are not Jordan measurable. So, let us see examples of sets in R, which are not Jordan measurable.

So, one of the easiest examples is to take E to be the rationals inside the interval 0 1. So, this set I claim is that is not Jordan measurable let us see why. So, if you consider the image Jordan measure for this set, E this is going to be 0, because no interval of strictly positive length is a subset of this set E, because it is only considers the rational points inside 0 1. So, the inner Jordan measure is going to be 0, what about the outer Jordan measure? So, the only interval which contains this set E is going to be bigger than 0 1.

And in fact, 0 1 the interval 0 1 closed interval is the smallest such interval which contains E. So, in fact, the outer Jordan measure is going to be 1 where we said that the inner Jordan measure is not equal to the outer Jordan measure, and this implies that E is not Jordan measurable.

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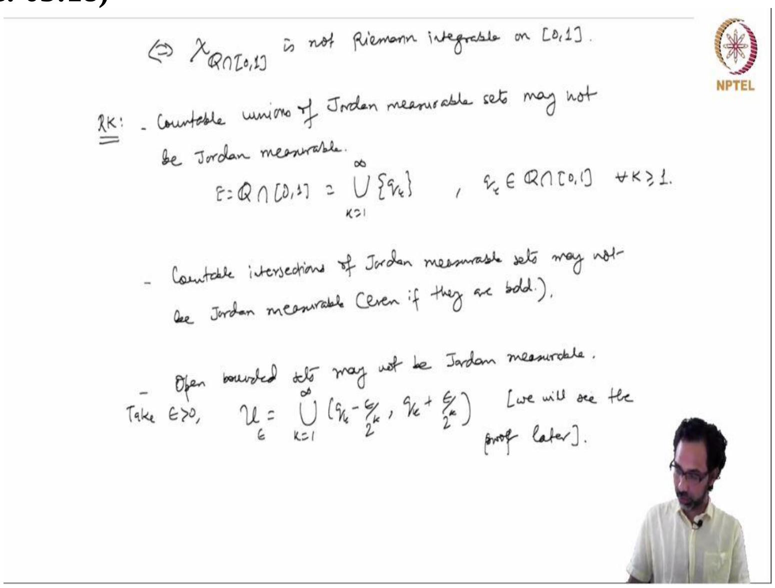

So, by connecting it to the Riemann integral, we can also say that equivalently that the indicative function for this set is not Riemann integrable on 0 1. So, this is the standard example of bounded function on 0 1, which is not Riemann integrable. Now, this example also shows so, I will put this as a remark it also shows that, countable unions of Jordan measurable sets may fail to be Jordan measurable.

So, here we can take each point each rational point QK and write this set E that we have as the union as a countable union of points which are rational numbers, so, QK belong to Q, Q intersection 0 1 for all K greater than or equal to 1. Of course, each single rational point is Jordan measurable with Jordan measures 0 this we already know. But in fact it is an elementary set. So it is Jordan measurable but when we take the countable union of each of these rational points in 0 1 then it fails to be Jordan measurable.

So, countable unions of Jordan measurable sets can fail to be Jordan measurement. Similarly, we will see later that countable intersections of Jordan measurable sets may not be Jordan measurable even if they are bounded so, there exists examples of bounded subsets of let us say R, which are countable intersections of Jordan measurable sets, but they are not Jordan measure.

So, we will see that the modification of the well known middle third cantor set will give us such an example, but we will come to that later.

So, I just put it as a remark for now, that countable intersection of Jordan measurable sets can also fail to be Jordan measurable another remark is that open bounded sets may not be Jordan measurable. So, one example for this is the following. So, take an epsilon greater than 0 and you can take the union of these intervals q k - epsilon / 2 to the power k and q k + epsilon / 2 to the power k. So, this set which depends on epsilon may fail to be Jordan measurable.

And we will see the proof later we will see the proof that this is not Jordan measurable later once we have defined the lebesgue outer measure it will be easy to prove that this set is not Jordan measure. So, this last example shows that even open sets may not be Jordan measurable and this is something that we want to have a nice theory of measure. So, that we are in contact with the topology of the underlying space. And so, we would like our measures, so, that at least the open sets that define the topology on our space, they should be measurable.

So, in this case this fits. So, all these arguments are just to motivate the lebesgue outer measure which will take care of all these problems, that countable unions and countable intersections of what we call lebesgue miserable sets, they will be lebesgue miserable and all open sets, whether they are bounded or unbounded they will also be lebesgue measurable. So, our lebesgue outrun measure we will take care of these problems that we face while using the Jordan measure and Jordan measurable sets.

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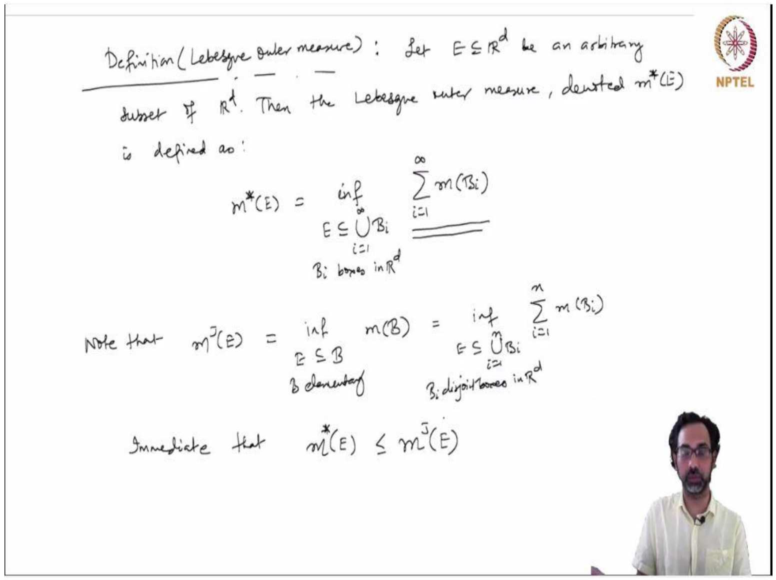

So, let us see the definition for the lebesgue outer measure. So, let E be a subset of R d an arbitrary subset, of R d then the lebesgue outer measures denoted by m upper star of E is defined as so, m upper star of E it is the infimum taken over covering of E by countably many boxes B i, so, B i boxes in R d and we take the following thing, we take the measure of each B i and sum it over the whole covering. So, we are taking infimum over these values given by the sum of from 1 to infinity.

The measure of B i where the union of these boxes B i will cover E. So, notice that for the Jordan outer measure, so, note that the Jordan outer measure is the infimum of E inside elementary sets B the elementary and we took the elementary measures of B, but now, each elementary measured each elementary set can be written as a finite union of B i where B i boxes in R d. And you take you can write the measure the elementary measure of B as the sum 1 to n m B i when we can take disjoint boxes so B i disjoint.

So, we see that the outer Jordan measured when we use the definition using finite boxes, if we replace this notion of finite boxes by infinitely countable infinitely many boxes B i and take the sum infinite sum of all the measures of B i then we pass from the outer Jordan measure to the lebesgue outer measure. So, this will allow us to treat countable unions and countable intersections and as well as open sets in R d satisfactorily. And from this definition, it is immediate that the lebesgue outer measured is bounded above by the Jordan outer measure.

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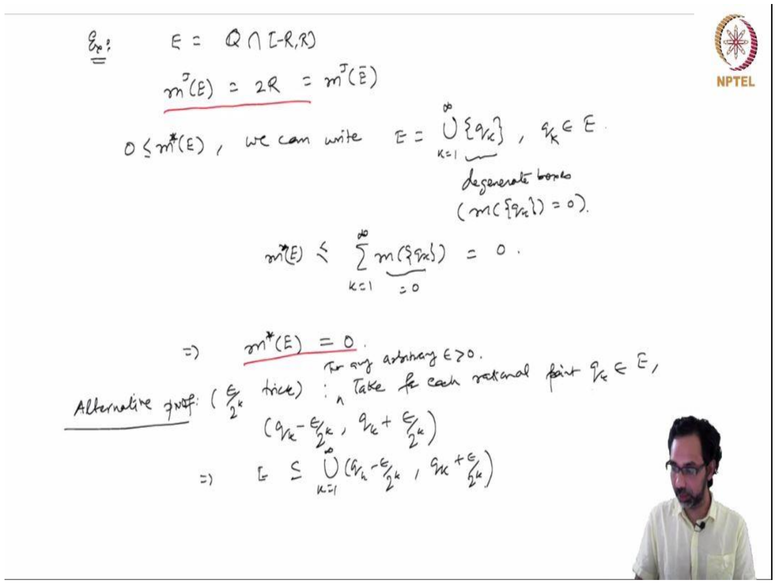

So, we see that the lebesgue outer measure is bounded above by the Jordan outer measure. But as the following example will show the lebesgue outer measure can be much smaller than the Jordan outer measure. So let us take again a which similar set then the last set we took for proving for a given a example of a non Jordan measurable set. So here I am taking E to be the rationals in the interval minus R to R. So the outer Jordan measure is of course, this is going to be 2R because this is the same as the closure of the set E.

The rationals are dense in this interval minus R to R and when you take the closer you get the whole interval minus R to R, so the length of integral interval is 2R. Now we will try to compute the lebesgue outer measure for the set. So we know that this is positive because it is only defined using measures of boxes. It is a sum of positive things, so it is always positive. So m star E is always positive. However, if you take if you write, we can write E as the union of this disjoint sets q k k from 1 to infinity q k belongs to E.

So, these are rational points within this interval minus R to R. Now, each of these can be regarded as a degenerate box, degenerate box meaning that its length is 0. So, the measure of this set q k this is equal to 0 and this is a covering of E itself. So, m star E is greater than is less than or equal to the sum of the sets q k k = 1 to infinity because this is a disjoint union and our m study was the infimum were all such union of boxes which cover E.

So, in particular this union of q k is a covering of E by boxes by diginet boxes. And so, m star is less than or equal to the sum of the measures of these things, but each one of them is 0. So, this sum is 0. So, therefore, we get that m star E is 0. So, here note that the Jordan outer measures can be very large depending on R capital R it can be very large, but, the lebesgue outer measure nevertheless turns out to be 0 to give an alternative proof of this same fact, without using degenerate boxes, what we can do.

So, this is an alternative I am giving this alternative proof because this technique will be used many times in this course. So, this is called epsilon / 2 to the n or 2 to the k trick. And what we can do here is cover take for each rational point q k inside E, we take the interval q k - epsilon / 2 to the power k q k + epsilon / 2 to the power k. So, this is for an arbitrary epsilon greater than 0 another E is a subset of the union of all these intervals q k - epsilon / 2 to the power k q k + epsilon / 2 to the power k.

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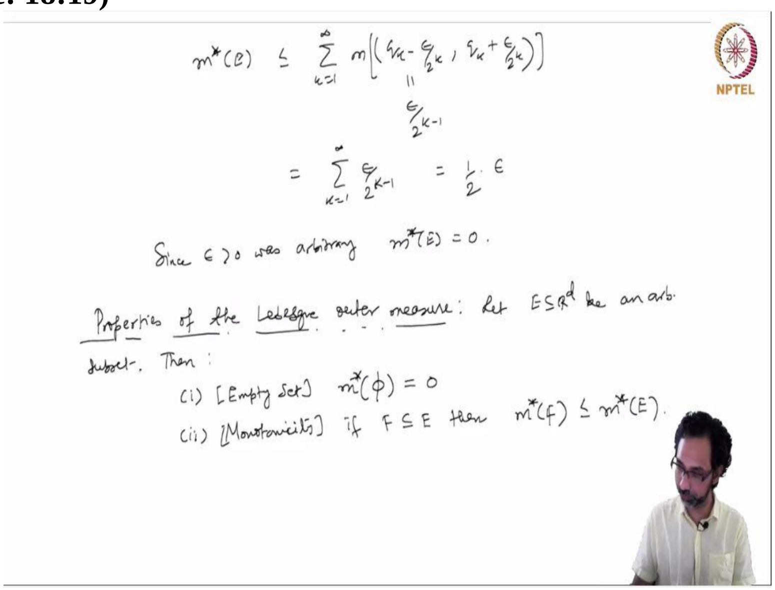

So, the lebesgue outer measure is less than or equal to the sum from k to 1 to infinity of these intervals q k - epsilon / 2 to the power k and q k + epsilon / 2 to the power k. So, now, we can easily compute what is this measure this is simply epsilon / 2 to the power k - 1. So, because each term can be explicitly computed, we can now easily compute the sum, this is the sum of epsilon over 2 to the power k - 1 and this is some constant c times epsilon.

Here c is actually half. So, since epsilon is arbitrary, we get that m star of E = 0. So, this trick of using epsilon over 2 to the power k so that the sum becomes finite is will be used many times in this course, and so it is worthwhile to understand how this is done. Now, let us come to some properties of the lebesgue outer measure, so, let E an arbitrary subset. So, here also there is another advantage that E can be unbounded for the Jordan measurable for the Jordan outer and inner measures.

We restricted ourselves to bounded subsets of R d, but here our subsets can be arbitrary. So, another thing that I should mention here is that the lebesgue outer measure in the definition of the lebesgue outer measure this can take the value plus infinity also. So, this value can be plus infinity, so, we allow infinite values for our lebesgue outer measure, in order to deal with unbounded sets. For example, if you take the entire R d, then one can show that this value is plus infinity. So, we will come to that later.

But, let us keep in mind that our lebesgue outer measure can be unbounded and can take the value plus infinity. So, now, for the properties, if you take E to be an arbitrary subset of R d then the first property is for the empty set this is this says that the lebesgue outer measure of the empty set E = 0. The second one is called monotonicity it says that if F is a subset of E then the measure of lebesgue outer measure of F is less than or equal to the lebesgue outer measure of E and the third one is called countable it may move to another page it is called countable sub additivity.

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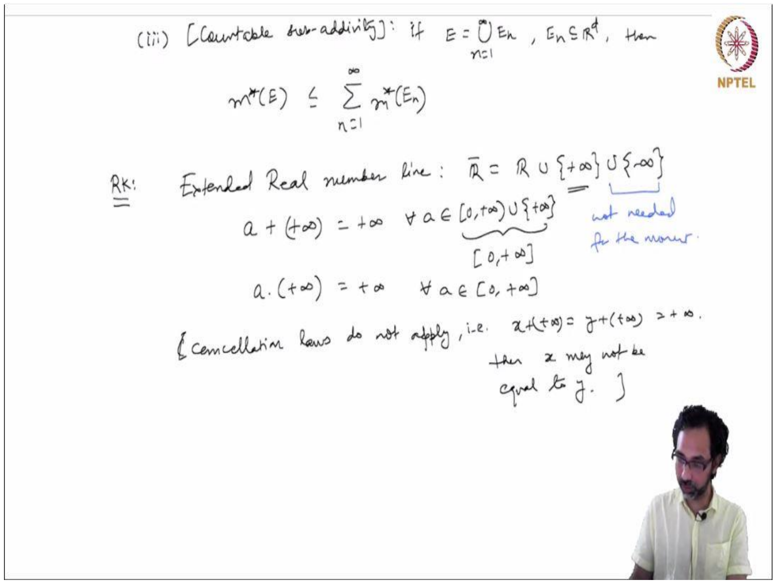

So, we have seen finite sub additivity but this one is upgraded to the countable unions of sets. So, if E can be written as a countable union of sets E n. So, E n are also subsets of R d then the measures the lebesgue outer measures of E is less than or equal to the sum given by summing up all these all the lebesgue outer measures for each E n. So, here remark is in order. So, note that since we are dealing with the possibility that m star E or each of these m star E ns can be can take the value plus infinity.

So, we have to define for example, what is plus infinity and plus plus infinity for example, and later on we will also see some factors here alpha n. So, then we have to define what is if m star E n is less infinity and alpha n is a real number then what is the product of such things. So, all these are in the structure which is called the extended real number line. So, this is denoted usually as R bar and this is R union 2 points which is plus infinity and minus infinity. So, since we are only dealing with positive numbers, so, for the moment we do not need this.

This is not needed for the moment and so, we only need to worry about arithmetic operations involving this symbol plus infinity, this is not a real number by definition, but we still need to define an arithmetic operations involving plus infinity. So for example, a plus plus infinity is equal to plus infinity for all a in this R union R plus. So, let me write 0 plus infinity, open union plus infinity, so we will write this set as 0 plus infinity closed.

Similarly, a multiplied by plus infinity is equal to plus infinity, for all a in 0 plus infinity, and so on I am not going to write out all the axioms, but you get the drift. What we expect to have if you add a positive number 2 plus infinity, you are supposed to get plus infinity, if you multiply a positive number with plus infinity, you will also get plus infinity and there are other rules for division and so on. You can look up these arithmetic operations in any book.

And one thing to note here is that cancellation laws do not apply. So, for example, if x plus plus infinity equals y plus plus infinity, then x may not be equal to y because the way we have defined it, both these are plus infinity. So, cancellation laws are not applicable always. So, we can only apply cancellation when you have finite numbers. So, we have to keep in mind these conventions for arithmetic in extended real number line. So, let us quickly prove these properties. So, the for the first one, I will leave the second one for as an exercise, this is an exercise and I will only prove first and third. So, for the first one, let us see the proof.

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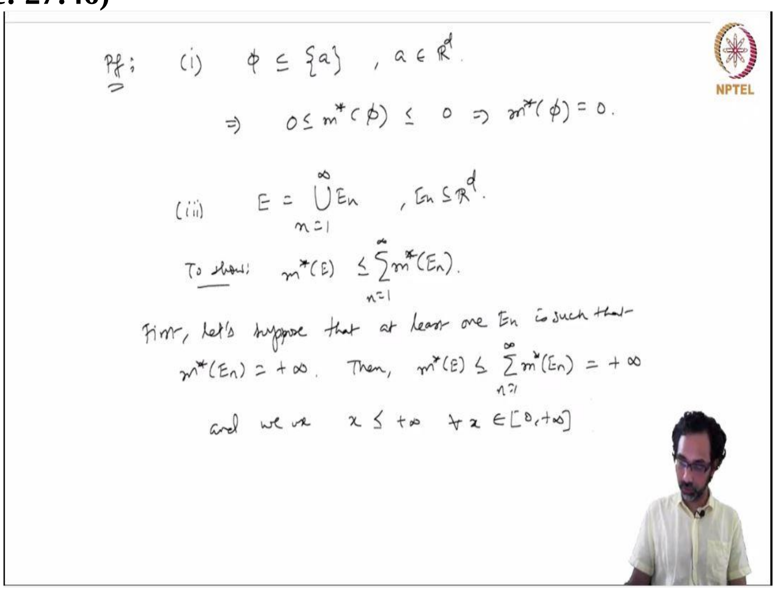

So, we can write phi as a subset of any real number or any point in R d. And because the single thinset has measured 0, this implies this is a covering by a degenerate box. So, we already knew that m star of phi is positive, but now, on the right hand side also this is 0. So, this implies that m start a phi is 0. And now, for the third one, so, here we had E a union of countably many subsets of R d and we had to prove that the lebesgue outer measure that the lebesgue outer measure of E is less than or equal to the sum from n equal to 1 to infinity m star of E n.

First let us suppose that at least one of these E ns suppose that at least one E n e such that the our lebesgue outer measure of E n is plus infinity then the above inequality holds by our conventions by our axioms of the extended real numbers, then m star E is always less than or equal to the sum from n equal to 1 to infinity m star E n because the right hand side is going to plus infinity and we use the fact that any real number x including any extended real number x is less than equal to plus infinity for all x in 0 plus infinity closed.

So, including plus infinity, we are imposing a partial order rather a complete order on this extended real number line, which stipulates that any real number is less than or equal to plus infinity including plus infinity itself. So, if at least one of the m star E n is plus infinity; then these inequalities satisfied by our conventions for the extended real life.

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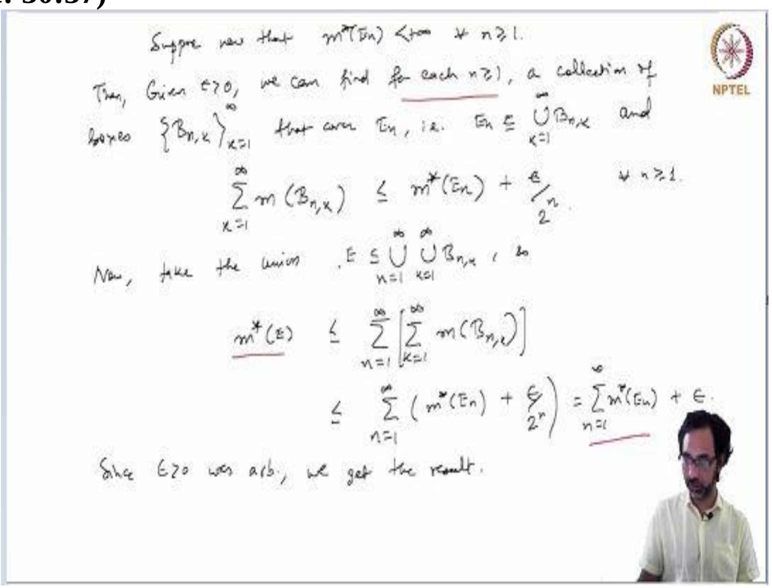

So, we suppose then that suppose now, that m star E n is finite for all n. So, then given epsilon greater than 0 we can find for each n greater than or equal to 1 collection of boxes countable collection B n m m from 1 to infinity. So, for each n this is for each n so, we are fixing n and covering taking a cover of boxes, which cover E n that cover E n. So, this means that E n is a subset of this union m from 1 to infinity B n m and not only that, it covers this E n we also want to use the infimum property.

So, we want to use that m star this sum from m to from 1 to infinity m star of B n m rather m there are too many m so, let me write k instead of m for the indices k B n k is and this is less than or equal to we want it to be close to the infimum value plus epsilon over 2 to the power n. And so, again we are going to use the epsilon over 2 to the power n trick, because in the end we want to sum on both sides. And so, we want to use epsilon over 2 to the power n rather than just absolutely.

So, now we take so, we have chosen for each n greater than or equal to 1, this collection B n k says that this inequality is satisfied for each end. So, this is where all in greater than or equal to 1. Now, take the collection union n = 1 to infinity union B n k k = 1 to infinity. So, this collection is rather this take the union this is a superset of E so, it covers E and this is a countable collection of boxes. So, m star of E is bounded above by this double sum n = 1 to infinity k = 1 to infinity measure of B n k.

So, this double sum one can take the inner sum and use the inequality that we have. So, we get n = 1 to infinity and then m star E n + epsilon / 2 to the power n and this last sum is equal to n = 1 to infinity m star E n + epsilon. So, on the left hand side, we have this outer lebesgue outer measure of E. And on the right hand side we have the sum of all these E ns lebesgue outer measures of E n plus an arbitrary constant epsilon. So, since epsilon is arbitrary you get the result. Inception was arbitrary we get the result that we wanted.

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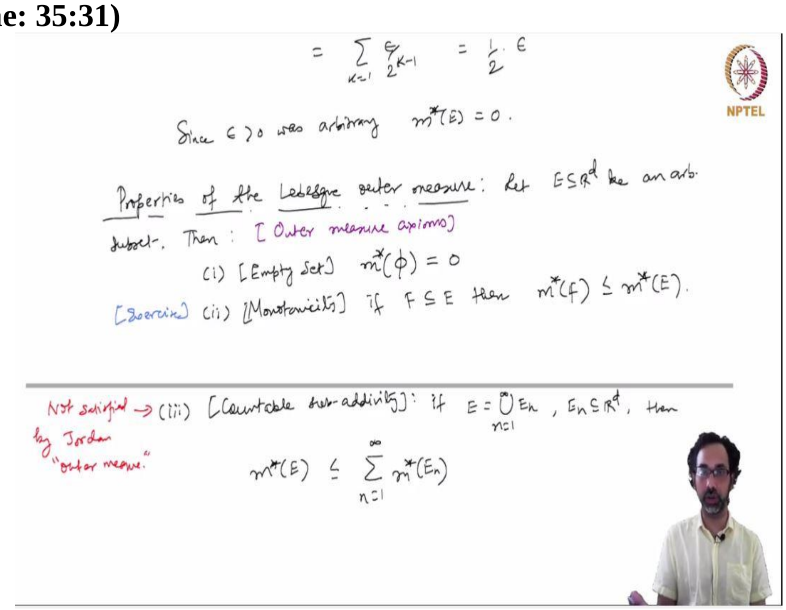

So, these 3 properties are together known as the outer measure axioms, outer measure axioms. When we are going to study abstract measure spaces, then we will model an abstract outer measured on an abstract space with these outer measure axioms, and so, the lebesgue outer measure is the prototypical example of an abstract outer measure. Note that the Jordan outer measure is not an outer measure with respect to these axioms because this countable sub additivity is not satisfied by the Jordan outer measures.

Because it is only satisfies finite sub additivity. So, in the sense, even though we are using the term Jordan outer measure, it is not it will not be an outer measure in the abstract sense when we define the concept of outer measures on abstract measures uses.