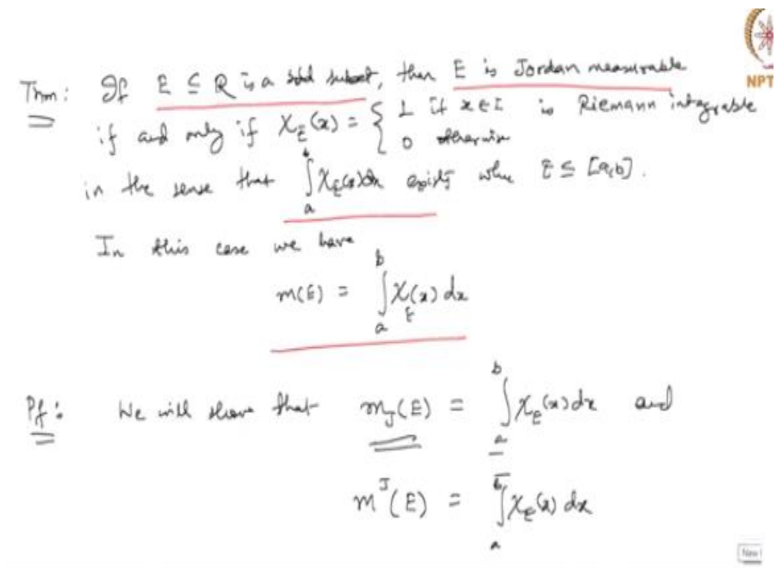
Measure Theory

Prof. Indrava Roy
Department of Mathematics
Institute of Mathematical Science

Module No # 04
Lecture No # 18
Connecting the Jordan Measure with the Riemann integral – Part 2

(Refer Slide Time: 00:22)

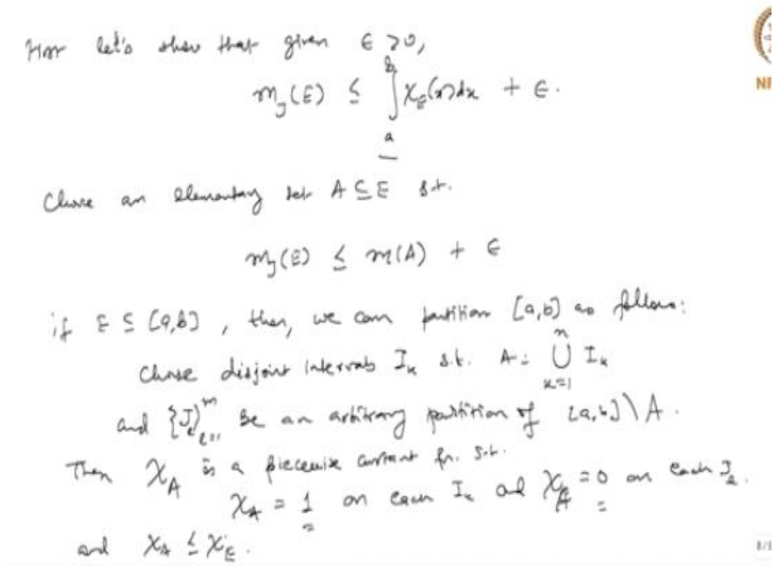


Now we return to the theorem that we want to prove which is stated here so we consider a bounded subset of R and the statement this theorem state that E this bounded subset E is Jordan measureable. If and only if the indicate function for this set E Ki E is Riemann integreable in the sense this integral exist and here I have added another statement which then identifies the Jordan measure with this integral from a to b of Ki e dx.

So, not only that Jordan measurability of E implies the Riemann integrability of the indicative function and vice versa. But the Jordan measure is actually exactly the value of integral of the indicative function. So let us see a proof here and we will show that the inner Jordan measure of E is precisely equal to the lower Darboux integral of Ki E. And the outer Jordan measure of E is precisely the upper Darboux integral of Ki E.

So I will just prove the statement for the inner Jordan measure and the proof for the outer Jordan measure is very similar.

(Refer Slide Time: 02:10)



So I will leave that as an exercise so first let us show that given epsilon greater than 0 the inner Jordan measure of E is less than or equal to the lower Darboux integral + epsilon. So since epsilon is arbitrary then it would imply that the inner Jordan measure is less than or equal to the lower Darboux integral. So to show this chose an elementary set A of E such that the inner Jordan measure is less than or equal to m of A + epsilon.

Now if E is the subset of this interval a, b then we can partition this interval a, b as follows so first choose intervals we can in fact choose a disjoint intervals Ik such that A is the union of Ik k = 1 to, n let us say. And we can and let JL from 1 to n be an arbitrary partition of the set a, b – A. So we take the partition inside A or disjoint it intervals Ik which is used in the disjoint union expression for the elementary set A and we can take an arbitrary partition for a, b – A.

Then the indicative function of A is a piecewise constant function such that Ki A equals 1 on each Ik and Ki A equals to 0 on each JL. So this is straight forward from the definition of the indicative function because if a point x belongs to one of this Ik's where it belongs to A and we will have the value 1. And if it is outside A then will have the value 0 so this is still a piecewise constant function and Ki of A is less than or equal to Ki of E. We have a point wise inequality of functions Ki A is less than or equal to Ki E.

(Refer Slide Time: 05:58)

therefore $m(A) = \sum_{k=1}^{n} m(I_k)$

$$= \sum_{k=1}^{n} 1 \cdot m(I_k) + \sum_{l=1}^{m} 0 \cdot m(J_l).$$

$$= \int_a^b \chi_A(x)\,dx$$

$$m(A) \leq \underline{\int_a^b} \chi_E(x)\,dx \quad (\text{since } \chi_A(x) \leq \chi_E(x) \ \forall x \in [a,b]).$$

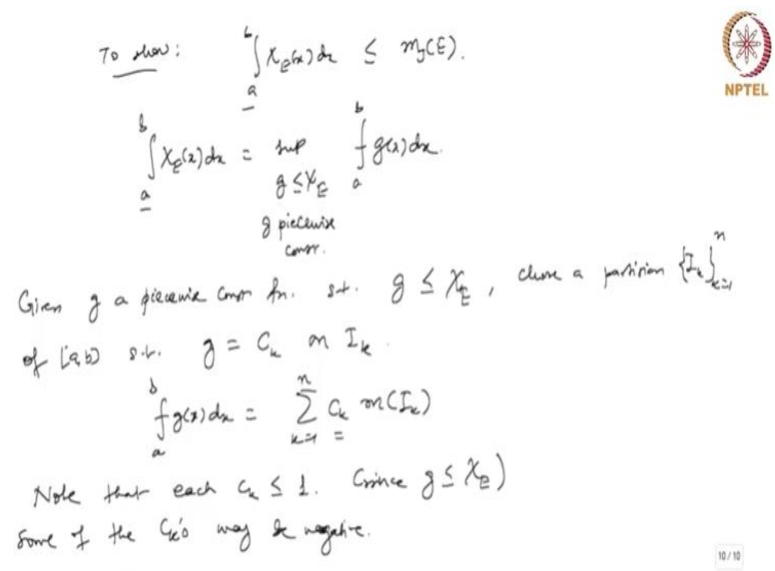
$$m_J(E) \leq m(A) + \epsilon \leq \underline{\int_a^b} \chi_E(x)\,dx + \epsilon.$$

Since $\epsilon > 0$ is arb. $\Rightarrow m_J(E) \leq \underline{\int_a^b} \chi_E(x)\,dx.$

So therefore m of A which is the definition of sum of mIk k = 1 to, n this can be written as k – 1 to, n 1 times nIk + L = 1 to, n 0 times n JL. So this is the piecewise constant Riemann Darboux integral for the indicative function of p which is the p square constant function. So we have indicated that the piecewise constant in Riemann Darboux integral for Ki A is nothing but the elementary measure of the elementary set A.

So therefore this is m of A is less than or equal to the lower Darboux integral from a to b of Ki e x dx. Since we also have the Ki a, x is less than or equal to Ki ex for all x in this interval a, b. So we have that the inner Jordan measure of E which was less than or equal to m A+ epsilon this is less than or equal to again the lower Darboux integral + epsilon. And since epsilon is arbitrary this implies that the lower the inner Jordan measure is less or equal to the lower Darboux integral of Ki E.

(Refer Slide Time: 08:12)



Now we have to prove the reverse in equality to show that the lower Darboux integral is less than or equal to the inner Jordan measure of E. So to show this remember that by definition we have this is the supremum of piece wise constant functions g which are point less than or equal to f, and you take the piece wise constant Riemann Darboux integral for the function gx now what we can do here is the following.

So if Jk so here I am using the fact that if given g a piece wise constant function such that g is less than or equal to. So this is supremum over Ki g less than or equal to Ki e if you take any piece wise constant function g such that g is less than or equal to Ki e because g is piece wise constant we can choose a partition Ik k = 1 to, n again say of this interval a, b such that g = ck on g takes the value Ck on Ik.

And in fact the piecewise constant Riemann Darboux integral for g is then Ck m Ik k = 1 to, n now in this sum we note that each Ck must be less than or equal to 1. Since g is less than or equal to Ki E and the indicator function takes the value at most 1. So each Ck is less than or equal to 1 also some of this Ck's may be negative. So sum of this Ck's may be negative so I will only consider those Ck's which are positive in this sum and then bound it above by 1.

(Refer Slide Time: 11:10)

if I' is the set of indices in $\{1, 2, \dots, n\}$ s.t.

$$1 \geq c_{i'} > 0 \quad \forall i' \in I',$$

$$\int_a^b g(x)\,dx = \sum_{k=1}^{n} c_k\, m(I_k) \leq \sum_{i' \in I'} m(I_{i'}) = m\Big(\underbrace{\bigcup_{i' \in I'}}\, I_{i'}\Big)$$

$\therefore$ the elementary set $\bigcup_{i' \in I'} I_{i'} \subseteq E$

(since $c_{i'} > 0 \ \forall i' \in I'$).

We can write: $\int_a^b g(x)\,dx \leq m\Big(\bigcup_{i' \in I'} I_{i'}\Big) \leq \underbrace{m_J(E)}_{\text{independent of } g}$.

$$\int_a^b \chi_E\,dx = \sup_{g \leq \chi_E} \int_a^b g(x)\,dx \leq m_J(E).$$

So therefore if Ck is positive so let me write the set of indices if J prime is the set of indices or let us say I prime is the set of indices in 1, 2 up to n such that Ci prime is greater than 0 for all i prime in this set i prime Which means that we are only choosing those indices for which this constant value for g is positive. So then the Riemann Darboux integral of gx which is equal to the sum k = 1 to n and Ck measure of Ik this is then less than or equal to the sum over I prime m of I i prime.

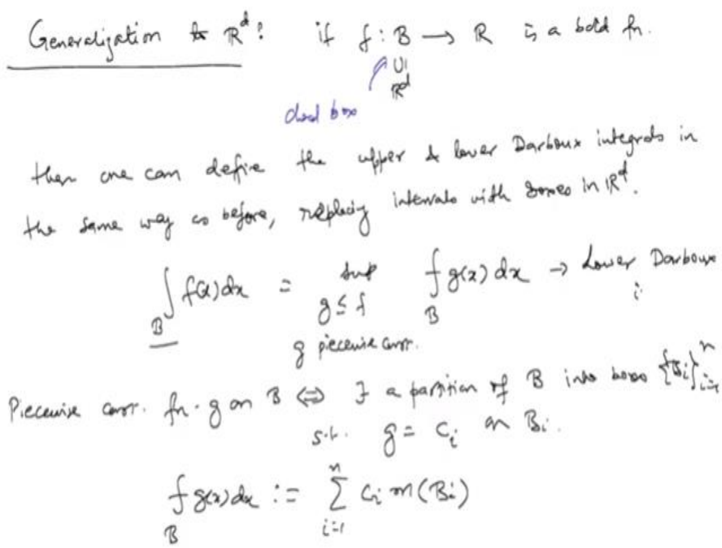
So we are only taking those indices for which Ci prime is greater than 0 but we already know that Ci prime must be bounded above by 1. And since we have discarded the negative values this can be bounded above in the sum Ck and mI k can be bounded above by this new sum over the indices for which C this value Ck are positive strictly positive. So now therefore they elementary set the union Ii prime belongs to the I prime this is an elementary sets that sits inside E that must sit inside E because Ci prime is greater than 0.

And therefore so this is because since Ci prime is greater than 0 for all i prime in i prime because otherwise this point wise inequality that we had for g and Ki will not hold. So we have produce an elementary set with sits inside E and the Riemann Darboux integral of g is bounded above by the elementary measure which is the elementary measure of the union i prime of Ii prime. So this is our required elementary set that we needed which sits inside E

And so we can write that the piecewise constant Riemann Darboux integral for g is less than or equal to the measure of the elementary set which is the union of Ii primes over i prime and this is less than or equal to the inner measure by the definition of the inner Jordan measure. So for each g for any g in fact which is piecewise constant on a, b we have this inequality which abounds the Riemann Darboux integral of the piecewise constant function g by the inner Jordan measure of g.

Now the right hand side of this inequality this is independent of g so therefore we can take the supremum over g less than equal to Ki E g piecewise constant and take the values of the Riemann Darboux integral of this piecewise constant functions. And this supremum will also be bounded above by the inner Jordan measure of E and this is nothing but on the left hand side this is nothing but by the definition of the lower Darboux integral this is the lower Darboux integral of Ki E. So we have the reverse inequality that we want to it.

(Refer Slide Time: 16:02)



Now we come to the generalization of the previous result to any Rd we have seen it for dimension 1 and we can easily generalize these results to arbitrary dimension. So for that if f is function from a box in Rd so this is the box closed box in Rd let us say. And it is supposed to be bounded function then one can define the upper and lower Darboux integrals in the same way as before replacing intervals with boxes in Rd.

So for example the lower Darboux integral over this box B of this function fx is equal to the supremum of over functions g which are point wise bounded by f g piecewise constant and we can take the piecewise constant Riemann Darboux integral of this piecewise constant function gx over this box B. So here piecewise constant function g on b is the same as saying that they are exist the partition of B into boxes Bi i = 1 to, n such that g takes the value Ci on Bi.

So it is constant on each Bi so we have replaced intervals with boxes and the rest of the definition remains the same and in this case the piecewise constant Riemann Darboux integral is by definition the sum Ci m of Bi i = 1 to, n. So we can also define in the same way the upper Darboux integral so this is the lower Darboux integral.

(Refer Slide Time: 19:15)

$$\overline{\int_B} f(x)\,dx = \inf_{\substack{f \le h \\ h \text{ piecewise} \\ \text{const. on } B}} \int_B h(x)\,dx.$$

Thm: If $E \subseteq \mathbb{R}^d$ is a bdd. subset, then E is Jordan measurable $\Leftrightarrow$ χ_E is Riemann-Darboux integrable

$\Leftrightarrow$ if $E \subseteq B$ then $\int_B \chi_E(x)\,dx$ exists

and in this case

$$m(E) = \int_B \chi_E(x)\,dx.$$

And we can also define the upper Darboux integral of fx over this box b which is the infimum of, f less than or equal to h piecewise constant m B and you can take the piecewise constant Riemann Darboux integral of the function h. So with these notions we can write the theorem which is the generalization of the previous theorem for arbitrary dimensions. If E is a subset as the bounded subset of Rd then E is Jordan measureable.

If and only if Ki E is Riemann Darboux integral which is the same as saying that if E is sitting inside a box b then the Riemann Darboux integral of Ki E x dx over this box B exist. And in this case the measure the Jordan measure of E is precisely this integral of the indicative function over B. So this is the straight forward generalization using the same proof basically we can rewrite the proof for arbitrary dimensions and it will work fine.

(Refer Slide Time: 21:18)

Corollary: If $f : [a,b] \to \mathbb{R}^+$ is a bdd. fn., then f is Riemann-Darboux integrable if and only if the subset $E_f \subseteq \mathbb{R}^2$ given by

$$E_f = \{ (x,t) \in \mathbb{R}^2 \mid x \in [a,b],\ 0 \le t \le f(x) \}$$

is Jordan measurable in $\mathbb{R}^2$. In this case

$$m(E_f) = \int_a^b f(x)\,dx$$

Pf: Carefully repeat the proof given for the statement $E \subseteq [a,b]$ is Jordan measurable $\Leftrightarrow \int_a^b \chi_E(x)\,dx$ exists.

So as a corollary as an application of this theorem or corollary we state the following result that if f is the function is a bounded function over an interval a, b then f is a Riemann darboux integrable. If and only if, the subset Ef of R2 given by this is similar to the region and the graph actually exactly the region and the graph which; you have seen before. Such that x belongs to a, b and 0 less than equal to t less than or equal to fx so here let me take only for positive real's R+.

So f only takes non negative values so this subset is Jordan measureable in R2 in this case the Jordan measure of Ef is exactly the Riemann integral of fx. So I will leave the proof as an exercise because it is essentially a repetition of the proof that was given earlier for the last theorem which stated that when he is Jordan measureable. If and only if the indicative function is Riemann Darboux integrable.

So I will just write that we have to carefully repeat the proof given for the statement E is the subset of a, b is Jordan measureable this was the statement of the theorem that was given before. If and only if the Riemann integral of the indicative function exist, so one has to carefully repeat it because there are some differences between that proof and this proof nevertheless the idea of the proof is entirely the same and one can replace the we can make the necessary replacement to get the proof for this case as well.