Measure Theory

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Module No # 04
Lecture No # 17
Connecting the Jordan Measure with the Riemann integral – Part 1

Now that we have seen many examples of Jordan measureable sets the last topic we will cover under Jordan measures is its de-connection with the theory of Riemann integration. So let us recall what is the definition of a recall integration of a function?

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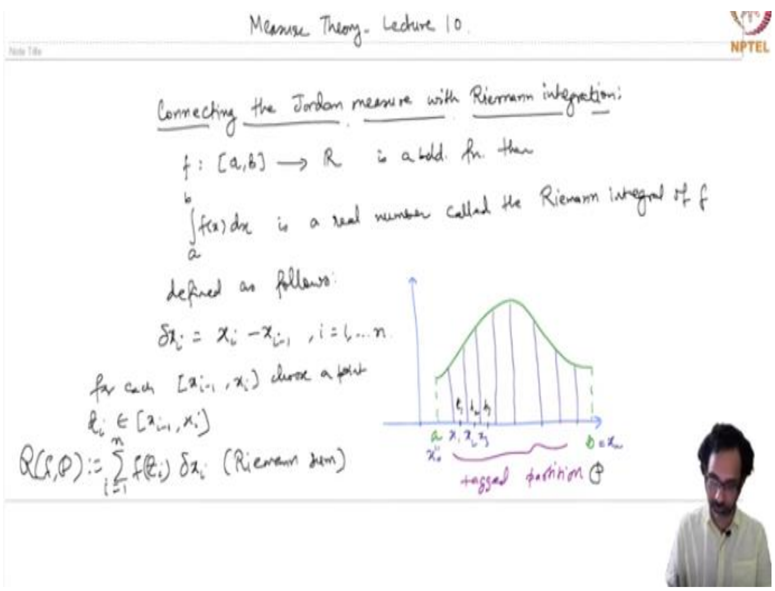

So if f is a function define on an interval a, b to say real number R and it is a bounded function then one can define the Riemann integration a to b fx dx is a real number called the Riemann integral of, f defined as follows. So I am going to just give you a brief recall of the original definition of the Riemann integral so the intuition is pretty straight forward. So if you have a function between points a, and b then you can partition this interval a, b into sub intervals of possible different length but that is not important.

So here you have points x naught which is equal to x1 then which is equal to a then x1, x2, x3 and so on. Then d = xm so you have n+1 points dividing the interval the segment a, b into sub intervals and we put delta xi to be the difference xi – xi-1. So i = 1 to, n here and then for each sub interval you choose a point call it t1, t2 and so on t3. So for each xi for each sub interval xi - 1 xi choose a point ti in this interval and in this way we found what is called tagged partition.

So this is called a tagged partition of A, b and then one defines the sum the Riemann sum is defined to be f of the value of f at this point ti multiplied by this length of interval del xi i from 1 to n. So this is the Riemann sum of, f with respect to the partition p that the tag partition p that have defined here. So this is called the Riemann sum of, f with respect to the tag partition p.

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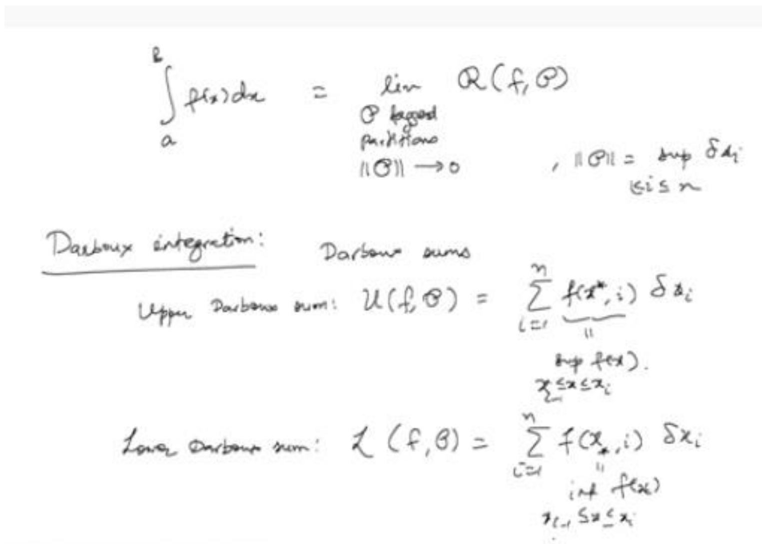

And the Riemann integral a to b fx dx is the limit of this Riemann sums fp the limit is taken over all partition P tagged partitions, and the maximum the norm of the partition goes to 0 where this norm of the partition is simply to the supremum of this del xi i = i from 1 to n. So as the size of the intervals in the partition goes to 0 you take the limit of this Riemann sums and if this limit exists then this is called the value that you get as the limit is called the Riemann integral of f

So this is original definition of Riemann but unfortunately if it is not very convenient in practice when you want to actually compute the integral the Riemann integral say a continuous function n over an integral then this tag partition definition is not very convenient to use. So an equivalent formulation is given by Darboux integration in this Darboux integral is defined using a similar partition but rather than using a tag partition it considers the maximum and minimum value.

So over this interval over a sub interval of the partition what is the maximum and minimum value of the function f that it takes on this sub interval. So here this lower part the lower horizontal line can be denoted as x lower star and f of x lower star rather and this is f of x upper star. So this is the supremum of the values of f over this sub interval and the lower star fx lower star is the infrimum of the values of f over this interval.

So the Darboux sums the Darboux integration is defined using the Darboux sums upper and lower Darboux sums. So the upper Darboux sums is defined uf with respect to partition p this is defined to be the sum from 1 to n fx I and the higher partition take the maximum value of f fx upper star and then multiply it with a size of the sub interval. So this is the supremum over xi less than equal to sorry xi – 1 less than equal to x less than equal to xi f of x is the supremum value of f over the sub interval.

And the lower Darboux sum is given by the same sum but instead of f upper star you use f lower star and this is the infrimum of fx x ranges between the interval xi – 1 to xi.

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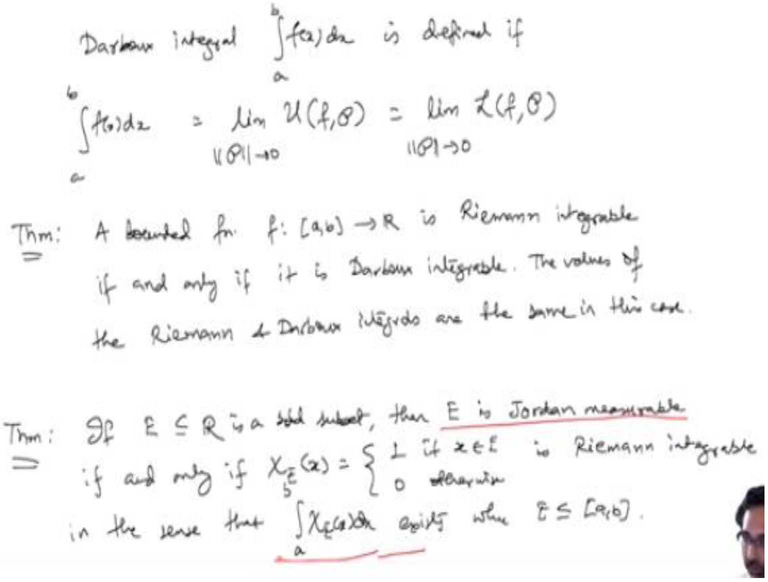

And the Darboux integral is defined integral f a to b fx dx is defined, if the limit as the norm of the partition goes to 0 ufb is equal to the limit as the norm of the partition goes to 0 as of the lower Darboux sum and this is equal to a to b fx dx. So these 2 limits are the same now fundamental theorem of integral calculus says that a function a bounded function f from an interval ab to R is Rieman integrable if an only if it is Darboux integral.

Now the definition due to Darboux is a more convenient to use and in fact not only so in fact this theorem not only says that when f is Riemann integrable if an only if it is Darboux integrable but also that values of the integrables are the same. So the values of the integrals of the Riemann and Darboux integrals are the same in this case. So one can use the Darboux definition and if; it more convenient to use as we will see.

So the connection with Jordan measure is a following theorem which is that if E a subset of R is a bounded subset. When E is Jordan measureable if and only if the indicative function of E Ki E which is the defined as 1 is x belongs to E and 0 otherwise is Riemann integerable in the sense that integral a to b Ki E x dx exist mean that it is a finite number where E is the subset of this interval a, b.

So because E is bounded subset we can always find a, and b such that a, is the subset of this interval a, b and in this case the indicative function from a, b to the real's is well defined. And then we can talk about the Riemann integral of the indicative function of E and this statement says that E is Jordan measureable if and only if this integral exist. So this is what we want to prove this is the first connection that we want to prove with the Jordan measure.

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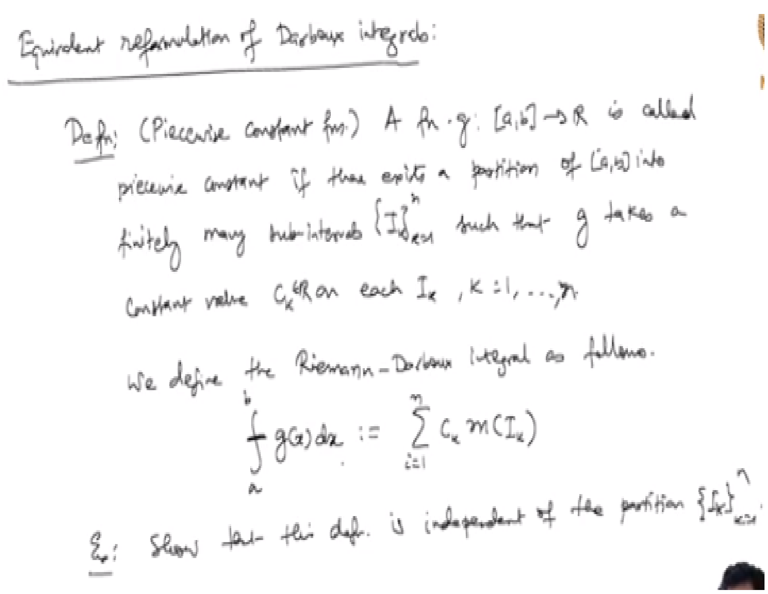

So let us trying to prove this but before we prove this we will make another equivalent reformulation of the Darboux integrals which will be more convenient for our purposes. And in fact it will give as an idea how to define the Lebesgue integral when we talk about Lebesgue measureable functions. So the equivalent reformulation of the above integrals depends on the definition of there so called piecewise constant functions.

So a function g from an interval a, b to R is called piecewise constant if there exists the partition of a, b into finitely many sub intervals is called an Ik k = 1 to, n. Such that this function g takes a constant value Ck on each Ik so Ck belongs to R on each Ik for k = 1 up to n. So this is piece wise constant function so for a piece wise constant function we define the Riemann Darboux integral as follows. So I will denote this as a, to b with a cut in between to show that to distinguish it from other function.

So the Riemann Darboux integral for a piecewise constant function of g is given by the sum i = 1 to, n ck m Ik. So this is a definition and the one should show as an exercise one should show that this definition is independent of the partition ik. So which means that if you have another partition I prime k then the values of g the constant value ck prime of g on those ik primes and will give you the same results when you sum up this ck mik's for any other partition .

So this is left as an exercise for you it is not a very difficult exercise to show so this is an unambiguous definition for the Riemann Draboux integral of a piecewise continuous a piecewise constant function.

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Define for a bdd fn. $f : [a,b] \to \mathbb{R}$

the upper Darboux integral

$$\overline{\int_a^b} f(x)\,dx := \inf_{\substack{f \le g \\ g \text{ piecewise const.}}} -\!\!\!\!\!\!\int_a^b g(x)\,dx$$

lower Darboux integral

$$\underline{\int_a^b} f(x)\,dx := \sup_{\substack{g \le f \\ g \text{ piecewise const.}}} -\!\!\!\!\!\!\int_a^b g(x)\,dx$$

it can shown: $\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx$

Now we define for a bounded function f from a, b to R the upper Darboux integral which is denoted a. b which is upper bar fx dx and this is defined as the infemum of functions which are point wise less than or equal to f where g is a piece wise constant function and you take the Riemann Darboux integral for the piecewise constant function g. Similarly the lower Darboux integral is denoted by a, b with lower bar fx dx.

And this is sorry the upper one is going to be f less than equal to g piecewise constant and the lower one will be given by g point wise less than equal to f g piecewise constant. And take the Riemann Darboux integral or the piecewise constant function g. So these are the upper and lower Darboux integrals and it can be shown without much difficulty again that the lower Darboux integral need to be fx dx is always less than or equal to the upper Darboux integral.

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when equality holds: $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx =: \int_a^b f(x)\,dx$

$$\underline{\int_a^b} f(x)\,dx = \lim_{\substack{P \text{ partitions} \\ \|P\| \to 0}} L(f,P)$$

$$\overline{\int_a^b} f(x)\,dx = \lim_{\substack{P \text{ partitions} \\ \|P\| \to 0}} U(f,P)$$

And when the equality holds in this inequality we call this the common value as the Darboux integral. So when equality holds this is the lower Darboux integral is equal to the upper Darboux integral then this common value is denoted by the integral of fx from a to b. And in fact one can also show that the lower Darboux integral is exactly this is the limit over all partitions with the norm of the partition is going to 0 of the lower darboux sums.

And the upper Darboux integral is precisely the limit over partitions with the norm of the partition is going to 0 of the upper darboux sums. So this definition is just a reformulation of our previous definition with respect to the upper and lower Darboux sums. But never the less this will be more convenient to use for our purposes.