Measure Theory
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Module No # 04
Lecture No # 16
Jordan Measure under Linear Transformations – Part 2

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$$\underbrace{m(L(E))}_{?} = C\, m(E).$$

Claim: if E is a closed box then $L(E)$ is a compact convex polytope.

Pf: if $H(c,v)$ is a half-space $(v \in \mathbb{R}^d, c \in \mathbb{R})$.

$$H(c,v) = \{x \in \mathbb{R}^d : x \cdot v \leq c\}$$

$$L(H(c,v)) = H(c, (L^{-1})^T v).$$

L takes half-spaces to half-spaces.

So m of L E must be a constant multiple you have written C times m of E so of course this also works in L as ranks strictly less than D but there we compute actually compute what is C which was 0. Here we will have to work a little bit more to compute what is C however there is a bit of a catch here because we do not know yet for an elementary set E this image L of E is Jordan measureable, and whether we can write m of LE because if LE is not Jordan measureable then m of LE does not make any sense.

So they main important part for the in this case is to show that L of E is in fact Jordan measureable. So for this n we will make this following claim that if E is a box then LE is a compact convex polytope. I will take a closed box here but it does not make sense much difference because you can take the closure of an open box and the Jordan measurability will not be affected.

So the claim is that if he is a closed box then L of E is a compact convex polytope so to show this so of course we know that the compact convex polytope is Jordan measureable. So we will finish our proof we will prove this claim for this case. So to show that LE is a compact convex polytope to note that if H c, v is the half space, so Hcv in our notation before it was the space of point in Rd such that the dot product of x with v is less than or equal to c.

So here v is an element of Rd and c is a real number so this was our half space which was given by x dot v less than or equal to c then the claim is that L of H cv is equal to the half space Hc L inverse transpose of v. So the linear transformation L takes half space to a half space here we are assuming that L is invertible so L inverse transpose make sense and this equality means that L takes half spaces to half spaces.

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if $x \in \mathbb{R}^d$ s.t. $x \cdot v \le c$ $\quad (x \in H(c,v))$

then
$$\underline{(Lx)} \cdot [(L^{-1})^T v] = x^T L^T (L^{-1})^T v$$
$$= x^T \underbrace{L^T (L^T)^{-1}}_{id} v$$
$$= x^T v = x \cdot v \le c$$

Let M be a compact convex polytope: $M = \bigcap_{i=1}^{n} H(c_i, v_i)$

$$L(M) = L\left(\bigcap_{i=1}^{n} H(c_i, v_i)\right) \underset{\substack{\uparrow \\ \text{invertibility} \\ \text{of } L}}{=} \bigcap_{i=1}^{n} L(H(c_i, v_i))$$
$$= \bigcap_{i=1}^{n} H(c_i, (L^{-1})^T v_i) = \text{compact convex polytope.}$$

So this is pretty straight forward because if you take if x is in Rd such that x dot v less than equal to c, then Lx dot L inverse transpose v this is equal to by definition of the inner product x transpose L transpose L inverse transpose v this is the same as x transpose L transpose L inverse transpose L transpose inverse v and this is identity. So this is x transpose v and this is just the dot product of x and v and this is less than or equal to c.

So the image of the half space which is given by L of x where x is in the half space H cv so here x belongs to Hcv Then L of x satisfies this inequality where Lx with the dot product L inverse transpose of e again less than or equal to c. So it takes a half space to a half space and now since a compact convex polytope let me call it M. Let M be a compact convex polytope then M can be written as the intersection of finitely many half spaces = 1 to, n H ci, vi.

So L of M is then L of this intersection Hci, vi and this is intersection of L Hci, vi and here I am again used the invertibility of L because in general this is not true it is only true injective. Because this intersection of the image is the image of intersection can only be true and L is injective. In this case L is in fact L is invertible so this equality holds and we know that this is for each I we have H L of Hci, vi is Hci L inverse transpose vi.

And this is again a compact convex polytope this is an intersect finitely many intersections of half spaces so this is a compact convex polytope. So infact L takes a compact convex polytope to a compact convex polytope and since a box in itself it is a compact convex polytope then the image of the box will again be a compact convex polytope. So we have shown that if he is a closed box then LE is a compact convex polytope. Now this shows that in fact if E is an elementary set then LE is Jordan measureable.

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$\Rightarrow$ if E is elementary in $\mathbb{R}^d$ then

$L(E)$ is Jordan measurable.

if $E = \bigcup_{i=1}^{m} B_i \Rightarrow L(E) = \bigcup_{i=1}^{m} L(B_i)$ (Jordan meas.)

$m(L(E)) = c\, m(E)$. $(c \geq 0)$.

(ii) To show: E is Jordan measurable then

$L(E)$ is Jordan measurable.

$m(L(E)) = c\, m(E)$.

So this implies that if E is elementary in Rd then L of E is Jordan measureable because if E is a disjoint union of box is Bi i = 1 to m say then L E is the union of LBi i = 1 to m and this is a finite union of compact complex polytopes. So this is Jordan measureable so in this way we have to prove that if E is an elementary set then L E is Jordan measureable and m of LE is c times m of E for some constant c greater than 0.

Well for the moment it is greater than equal to 0 because we do not know whether c 0 or not but we will find out that if determinant of if L is not. If L is invertible then determinant of L is not 0 and c is just it should be the modulus of the determinant. So we will prove that later so this proves the first part of the theory the second parts is that is so this is what we want to prove that if E is Jordan measureable then LE is Jordan measureable.

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Let $\epsilon > 0$. Chose an elementary set $A \subseteq E$ such that

$$m_J(E) \leq m(A) + \epsilon$$

Then $L(A) \subseteq L(E)$

so $m(L(A)) \leq m_J(L(E))$

$$\Leftrightarrow \quad c\, m(A) \leq m_J(L(E))$$

$$\Rightarrow \quad c\,(m_J(E) - \epsilon) \leq m_J(L(E))$$

$$\Rightarrow \quad c\, m(E) - c\epsilon \leq m_J(L(E))$$

Similarly, one can show: $m^J(L(E)) \leq c\, m(E) + c\epsilon$.

$$c\, m(E) - c\epsilon \leq m_J(L(E)) \leq m^J(L(E)) \leq c\, m(E) + c\epsilon.$$

And again the same relation holds that m of LE = c times m of E so to show this let epsilon greater than 0 be given. And we chose an elementary set A, a subset of E such that mJ lower subscript J which means you are taking the Jordan inner measure of E is less than or equal to mA + epsilon. Then the image of A is the subset of the image of E and so the elementary measure the Jordan measure of LA is no longer elementary.

So we can only talk about the Jordan measure because it is compact convex polytope or a union of compact convex polytopes never the less it is Jordon measureable. But m of LA is less than or equal to the inner Jordan measure of LE simply by monotonicity. But we know that m of LA is c times of m of A because A is elementary and so this is less than or equal to the inner Jordan measure of LE.

And this implies that because mA is greater than or equal to mJ the inner Jordan measure of E minus epsilon. So the inner Jordan measure of E minus epsilon is less than or equal to the inner

Jordan measure of LE which means that c mJ are actually it is just m of E because E is Jordan measureable – c epsilon is less than or equal to the inner Jordan measure of LE. Similarly one can show using the outer Jordan measure that the outer Jordan measure of LE is less than or equal to c mE + c epsilon.

So we will get a chain of inequalities c mE – c epsilon is less than or equal to the inner Jordan measure of LE is less than or equal to the outer Jordan measure of LE and this less than or equal to c mE + e epsilon. Now since c is a constant independent of E it is independent of epsilon which means that c epsilon is arbitrary.

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$$\Rightarrow \quad m_J(L(E)) = m^J(L(E)) = c\,m(E).$$

(iii) To show: $c = |\det L|$.

Fact: if L is invertible then it can be written as $L = UP$

where U is an orthogonal matrix

P is a positive-definite (real symmetric) matrix.

L_1, L_2 are linear transformations, then

$$\underbrace{m((L_1 L_2)(E)) = c_1 c_2\, m(E)}_{\text{Check this claim.}} \quad \text{where} \quad \left\{\begin{aligned} m(L_1(E)) &= c_1\, m(E) \\ m(L_2(E)) &= c_2\, m(E) \end{aligned}\right\}$$

And therefore this implies that m the inner Jordan measure of LE is equal to the outer Jordan measure of LE is equal to c of m E. So in particular these 2 things are equal and LE is Jordan measurable we have proved in the same statement that it is equal to c times m of E where c is the same constant that was used in part 1. Lastly we have to show that the third part claim that c is equal to the modulus of the determinant of L.

So for this we again need some facts from linear algebra which is that if this is the fact if L is invertible then it can written as L equals u times P where u is an orthogonal matrix and P is a positive definite. So this is in particular real symmetric matrix so we can use this factorization to prove that c is in fact equal to the modulus of the determinant of L. So first if u okay so we will use this fact that m of L1, L2 of E so if L1, L2 are linear transformations.

Then if L1 and L2 are linear transformation then m of L1, L2 E so the multiplication of L1, L2 matrix multiplication this is c1 c2 m of E where m of L1 E equals c1 m E and m of L2E equals c2 m E. So this can be very easily checked so check this claim and it follows straight forward from these 2 facts.

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$$|\det L| = |\underbrace{(\det U)}_{=1}| \cdot |(\det P)|$$

$P = Q^T D Q$ where Q is orthogonal matrix
D is a diagonal matrix.

Check that: $m(D(E)) = |(\lambda_1 \lambda_2 \cdots \lambda_d)|\, m(E)$

where $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_d \end{bmatrix}$

$$c = m'([0,1]^d) = m(D([0,1]^d)) = |\lambda_1 \lambda_2 \cdots \lambda_d|$$

So if L = u multiplied by P then we have determinant of L is determinant of u multiplied by determinant of P and the modulus is multiplication of the modulus. But the modulus of the determinant of an orthogonal matrix is equal to 1 and so we are only left with this part the predominant of p where p is a positive definite real symmetric matrix. Now we can further decompose p as Q transpose D Q where Q is again an orthogonal matrix and D is a diagonal matrix.

So this is about diagonalization of invertible real symmetric matrix of course the L is invertible then P is also invertible in this so called polar decomposition. So this is called this is the polar decomposition so again we are left with an orthogonal matrix Q and the diagonal matrix D. So it is enough to check the formula for c when your linear transformation is an orthogonal matrix in which case c should give you 1 and when it is a diagonal real diagonal matrix in which case it should give you the multiplication of the diagonal limits and taking the modulus of it .

So I will leave the diagonal part as an exercise so check that m of D E is equal to the multiplication lambda 1, lambda 2, lambda d modulus m of E where D is the diagonal matrix lambda 1, lambda 2, lambda d and other entries are 0. So remember that this constant c was m prime when we define another map on the elementary subsets of Rd to the positive real’s. Then this was simply the measure of the unit box D dimensional unit box 0, 1 to the d.

So this in our case we had the multiplication we have the linear transformation L so we just have to compute the measure of the image of the D dimensional unit Q under the linear transformation L. So it is quite easy to compute what is the measure in case D is the diagonal matrix lambda 1, lambda 2 up to lambda d because it will scale each coordinate with the lambda I. So then we will have the modulus of the product lambda 1, lambda 2 up to lambda d.

It will still be a box so in this case when D is diagonal matrix then the image will still be a box. So here L is our D and in this case this will be simply modulus of Lambda 1, Lambda 2 Lambda d.

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in case L is an orthogonal matrix.

Take $E = \overline{B(0,1)}$ (Since L preserves dot products,

we have $L(E) = E$

$m(L(E)) = 1.m(E)$

$|\det L| = 1$

Now in case is an orthogonal matrix then take E to be the unit ball close unit ball center at 0 with radius 1. So in this case an orthogonal matrix preserves the unit ball because L is an orthogonal matrix of Rd and it preserves the dot products. So since L preserves dot products we have L of E L of the closed unit ball is the same as E itself. So then we have m of LE is equal to m of E and this can be written as 1 times m of E.

And of course 1 here is the determinant of L because the modulus of determinant f L because L is an orthogonal transformation. So in this case we can also check that or formula holds that c is the modulus of the determinant of our linear transformation and this will conclude our proof because once we have checked it for orthogonal matrixes and for real symmetric matrixes then our polar decomposition will ensure that the constant c is given by the determinant of L where we can use now the multiplicativity of the determinant.