# Measure Theory

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**Module No # 02**
**Lecture No # 15**
**Jordan Measure under Linear Transformations - Part 1**

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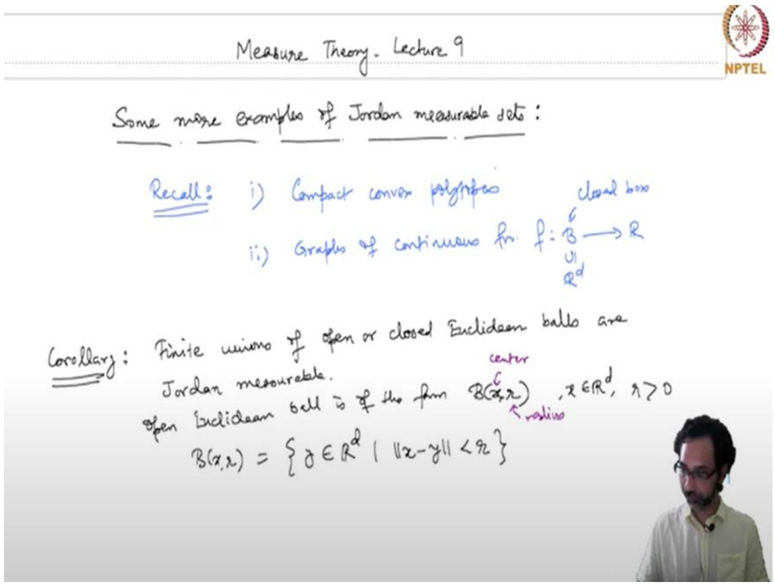


In this lecture we will look at some more examples of Jordan measurable sets. We have already developed quite bit of theory for Jordan measurable sets and in the last class we saw 2 examples which gave a wide class of Jordan measurable sets. So lets us recall that the first 1 was about compact convex polytopes and the second one was graphs of continuous real valid functions from a closed box B to R where B is the subset of Rd so this is a closed box in Rd.

So as a corollary for the second one we can establish some more examples of Jordan measurable sets. So as a corollary we can state that finite unions of open or closed Euclidean balls are Jordan measurable. So an open Euclidean ball is of the form B x r we will denote by B x r an open Euclidean ball. So the center will be this x this is the center of the Euclidean ball and r will be the radius of the Euclidean ball. So here x belongs to Rd and R is a positive real number.

And this B x r is the set Y belonging to Rd such that the Euclidean distance between x and y is strictly less than R. So this gives you the open Euclidean ball of radius R with center x.

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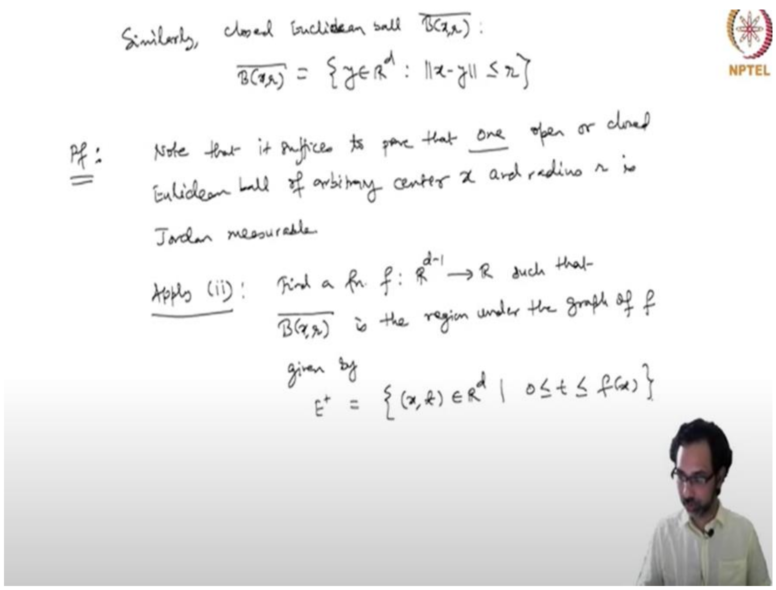


Similarly a closed Euclidean ball which is simply the closure of such open Euclidean ball B x r and they are of the form. So the closure is simply the sets of points in Rd such that the Euclidean distance between the x and y is less than or equal to r. So one can check that in fact when you put less than equal to in place of strictly less than sign then this set becomes the closure of the open Euclidean box.

So how do we prove this that these are Jordan measurable? So note that it suffices to prove that 1 open or closed Euclidean ball of arbitrary center and radius center and arbitrary center let say x and radius r is Jordan measurable because we know that finite union of Jordan measurable sets is Jordan measurable. So we are finding union of such open or closed Euclidean balls and each one is Jordan measurable. So the finite union will be Jordan measurable.

So to prove this we can simply apply the second part of the recall I mention before this is that graphs of continuous functions are Jordan measurable. So we have to find a function f from Rd - 1 to R such that the closed Euclidean ball is the region under the graph of x. And this is given by this kind of sets that are denoted E+ before this is simply the set of points x t in Rd such that 0 less than or equal to t less than or equal to fx.

So we have to find such a function and we can easily find one by just taking the equation of sphere a d sphere in Rd.

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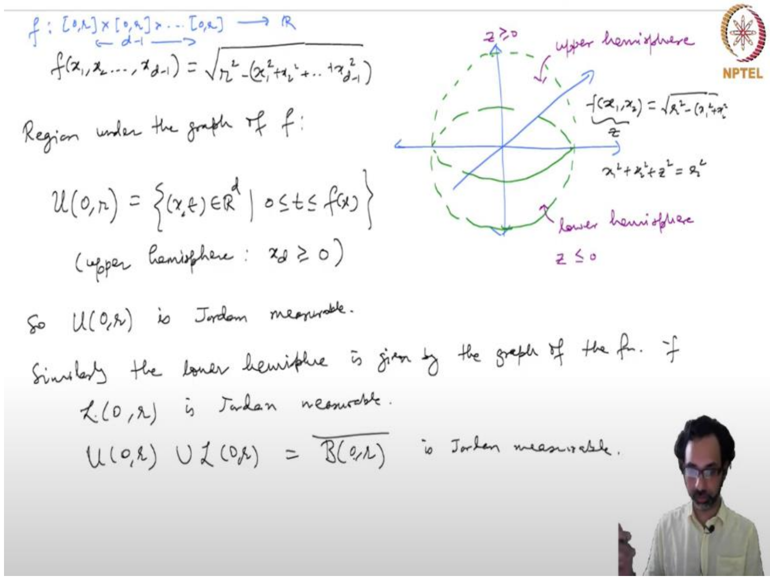


So for example if you have 2 spheres in R R3 so here we have R3 and we have the 2 spheres here we have 2 spheres we can separate this sphere into their upper hemisphere. So this is the upper hemisphere for which the z coordinate is greater than equal to 0 and this is the lower hemisphere lower hemisphere for which the Z coordinate is less than equal to 0. So in this way we can write the closed Euclidean ball as a union of 2 things and the region under which under each of these spheres or hemisphere becomes half of the Euclidean ball.

So in this way we can take the union of these 2 halves to produce our closed Euclidean ball. So our function should be simply be given by x1, x2, xd - 1 so because it is from Rd - 1 to R. And this is simply the square root the positive square root of the function R square - x1 square x2 square + x d - 1 square. So for example here f x1, x2 where simply be R square - x1 square + x2 square.

So that if you turn this as z or x3 then x1 square + x2 square + z square is R square and this is the equation of sphere in their dimensions a two sphere in 3 dimensions. So here we can take the function f to be the simply given by the equation of this upper hemisphere. And then the region under the graph of, f which is let me write this as upper hemisphere which centers 0 and radius r this is given by x t belonging to Rd such that 0 less than or equal to t less than or equal to f of x.

This is the upper hemisphere and this corresponds to fact that xd greater than equal to 0 for this part. Here as we have z greater than equal to z direction. Here we can take the last coordinate xd to be greater than equal to 0 and because of this the region under the graph so u0 r is Jordan measurable. Similarly the lower hemisphere is given by the graph of the function -h. By the way here we have to restrict our x1 x2 - by d - 1 to a closed box.

And here the box is so the f is define so let me write it here f is define on the Cartesian product of 0 r. So this there are d - 1 coordinates and it takes values in R ok. So here rather than F if it will take minus f it will get a lower hemisphere which I also denote as L0 r and this is also is Jordan measurable. So the union of L0r and u0 r, L0 r is simply the closed Euclidean ball which center 0 and radius r and this is Jordan measurable.

And the general case for when you want to replace the center you can have arbitrary center x rather than 0 can be simply achieved by translation invariance of Jordan measurable sets.

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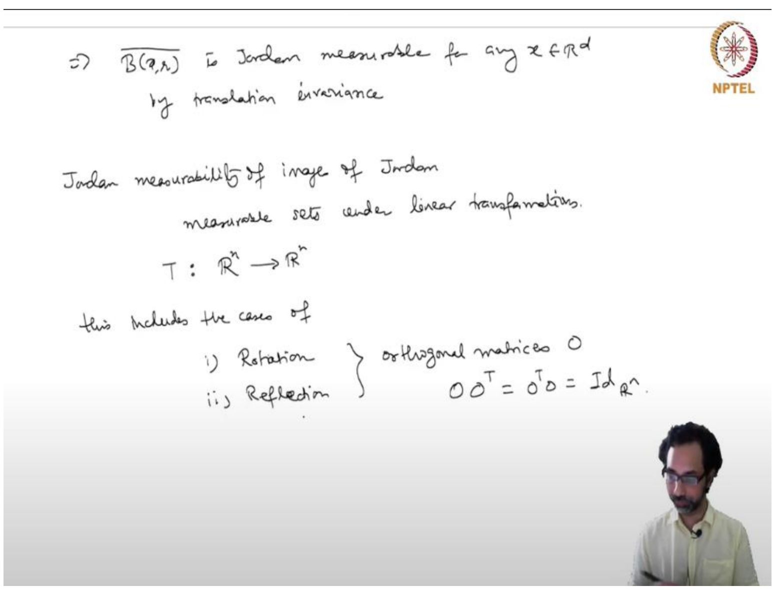


So this implies that b x r is Jordan measurable for any x in Rd by translation invariance. While translation invariance I mean that when you translate by x then the Jordan measurability is invariance. So in that sense it is translation invariant it is also translation invariant in terms of Jordan measure. So the Jordan measure of B 0 r closure B x r closure is also same. So in this we can see that finite unions of open and closed Euclidean balls are Jordan measurable. The next set of examples comes from a linear transformation.

So once we know that a set is Jordan measurable if we apply a linear transformation on that set and look at the image we can ask the question whether the image set is Jordan measurable or not. So this is about Jordan measurability of images of Jordan measurable sets under linear transformations. So by linear transformation I mean matrix T from Rn to Rn so it is a given by an n cross n matrix.

And this includes the cases this includes the cases of 1 rotation and 2 reflection and both are given by the set of orthogonal matrices O. So orthogonal matrix let us recall that and matrix is orthogonal if O, O transpose is O transpose O and this identity matrix on Rn. So we will look at rotation of Jordan measurable sets and reflections on Jordan measurable sets and if you recall our original goal to define the Jordan measure was to have the invariance of the measure under rotations and reflections and translations.

So we have already seen that the Jordan measurable is translation invariant and next we will see that under rotation and reflection and in particular for any orthogonal matrix the first of all the image is Jordan measurable. Second, of all that the measure remains the same when you consider the image of the Jordan measurable set and the rotation and reflections. So I am going to state it as a theorem.

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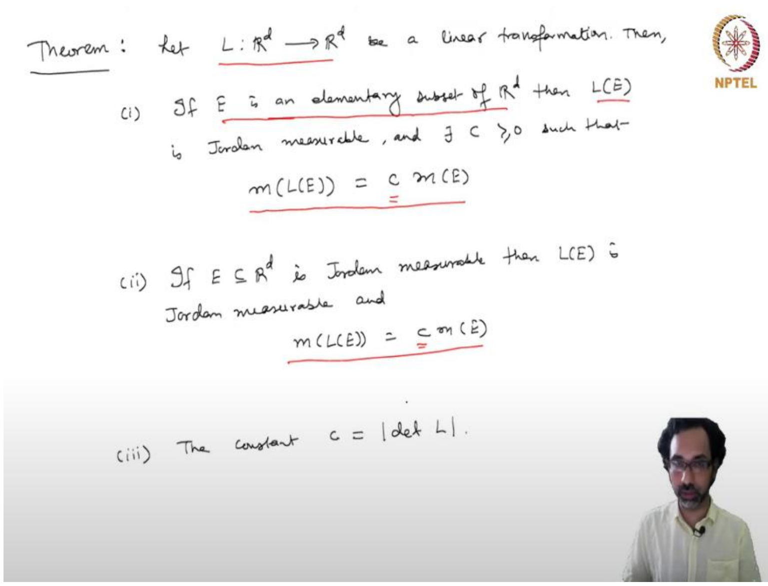

So let us look at the first part of the theorem so I am considering L from Rd to Rd a linear transformation then the first part says that if E is an elementary subset of Rd then the image LE is also a Jordan measurable. And then exists a constant c a positive constant c such that the measure of the image is c times the measure of E. The second part says that if E is not Jordan measurable set of subset of Rd no longer an elementary substance even then the image L of E is Jordan measurable.

And this same equation as in the case of elementary sets continuous to hold which is that the Jordan measurable L E is c times of m of E. This c is the same c in part 1. And the third part of the theorem says that the constant c can be explicitly determined and it is simply the modules of the determinant of the linear transformation L. In fact we can allow c to be equal to 0 as well. So this constant which is given by modulus of the determinant of this linear transformation L can in fact take the value 0.

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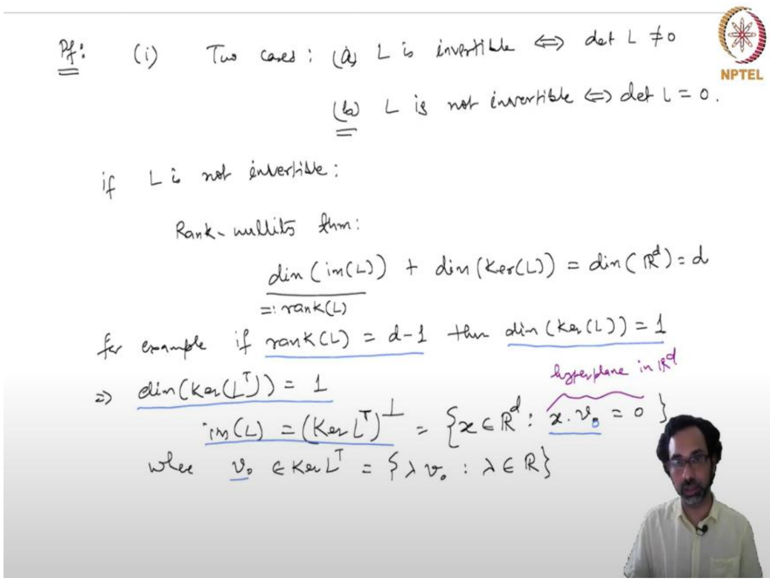


Now the proof is a bit long and we would require several steps. And we will also need some facts from linear algebra. So for the first one which is about the images of elementary set it is helpful to divide the problem in 2 cases. The first one is if a so first one from this case is when L is invertible and the second one when L is non-invertible. So remember that this is equivalent to saying that if you wanted to be invertible then determinant of L must not be 0.

And similarly if it is not invertible then the determinant of L is 0. So in fact for the first one if we see the third one should imply that the c should be 0 and the measure of the image should be 0 and this what we expect. And this is for the second case when L is not invertible because the determinant will be 0. So let us treat this case first so if L is not invertible then what happens? So in this case we recall the rank nullity theorem.

Let says the dimensions of the image of L + the dimensions of the kernel of L is the dimension of Rd which his simply d. So if L is not invertible then the kernel has dimensions strictly greater than 0. So for example if so the dimension of the image is called as rank of L this is equal to rank by L of definition. So if the rank of L = d - 1 then the dimension of the kernel of L is 1 and one can also reduce that the dimension is implies that the dimension of the kernel of L transpose is also equal to 1.

So in this case the image of L is simply the perpendicular of the kernel of L transpose which is the set x in Rd such that x dot v naught = 0. Where v naught belongs to kernel of L transpose. And because kernel of L transpose is 1 dimensional this is the kernel of L transpose is simply the scalar multiplication of v naught. So such that Lambda is in R so because we have rank, of L is d - 1 then the dimension of kernel is 1 and so the dimension of kernel transpose is also 1.

So it is generated we can take as bases some vector v naught in kernel of L transpose and any element of kernel of L transpose will be given by lambda times v naught. And so the image is the perpendicular of kernel of L transpose and this is simply given by the set of points in Rd such that the x inner product of x and v naught is 0. So note that this is the hyper plane equation in Rd. And we already know that in hyper planes have measures 0.

First of all they are Jordan measurable and they have measure 0 once you have the bounded segment of the hyper plane.

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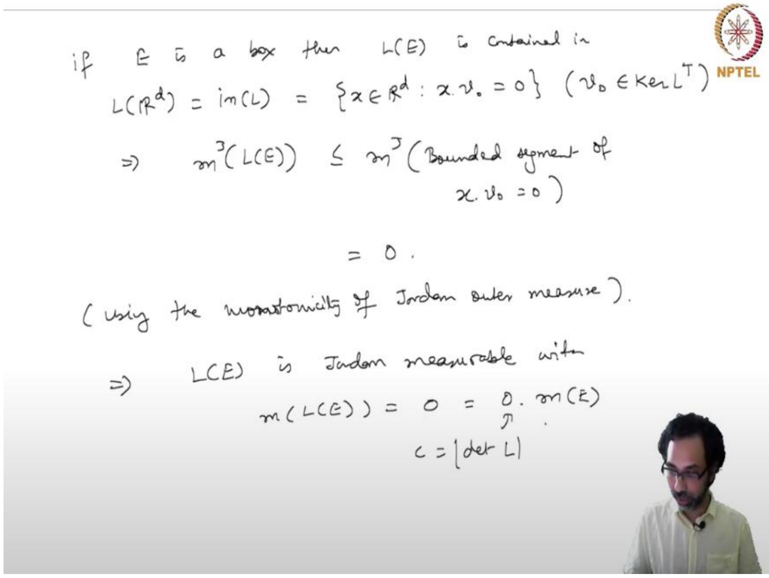


So if E is a box then L E is contain in L Rd which is the image of L and this is this is given by the hyper plane x in Rd such that x dot v naught = 0 where v naught belongs to the kernel log L transpose. So L E is contained in hyper plane and this implies that the outer measure of L E is less than equal to the outer measure of some bounded segment of the hyper plane x dot v naught = 0 and this is 0.

Here I am using the monotonicity for the Jordan outer measure so this is using the monotonicity of Jordan outer measure which is immediate from the definition of the Jordan measure. So we see that the Jordan outer measure for the image of a box E is the box here under the linear transformation L has Jordan outer measure 0 which implies that L E is Jordon outer measure Jordon measurable with Jordan measure 0 with m of L E = 0.

And this can be written as 0 times m of E and this will be our determinant of L because L has a rank strictly less than d so the determinant is going to be 0 I am sorry the determinant of L going to 0. Hence so in this case c is determinant of the modulus of determinant of L which is 0. So we see that this equation is satisfied.

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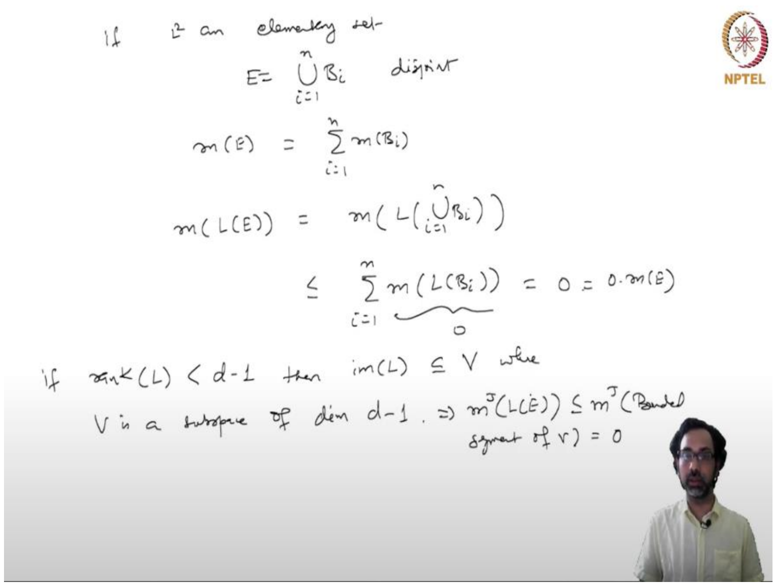


And now we can if you have an elementary set E if E is an elementary set then E can be written as union of boxes Bi which are disjoint from each other. And then m of E is given by sum of m of Bi i = 1 to n and similarly the measure of L E when is going to be less than is equal to the measure of L union Bi i = 1 to n which by finite sub additivity will give you the some of m L Bi i = 1 to n but each of these is 0 so this is 0.

Therefore, again this can be written as 0 times m of E. So this is again satisfied when E is an elementary set and L has rank strictly less than d in this case we have taken rank d - 1 here we have taken d - 1. But if it has even lesser than d - 1 then include it could be included in a in some space of rank dimension d - 1 and still it will be a subset of a set which has Jordan measure 0. So we have Jordan measure 0.

So here if rank of L is less than d - 1 then image of L can be included in a set in a subspace let me call it V where V is a subspace of dimension d - 1. And this implies that the outer measure of image of L is L of E is less than or equal to the outer measure of bounded segment of v which is going to be 0 because v is the subspace of dimension d - 1. So again we see that if you take any rank less than d - 1 then again the outer measure is going to be 0 we can repeat the same argument to prove that L E is Jordan measure.

So we have proved that if E is an elementary set and L has rank less than d strictly less than d then L E is Jordan measurable with Jordan measure 0.

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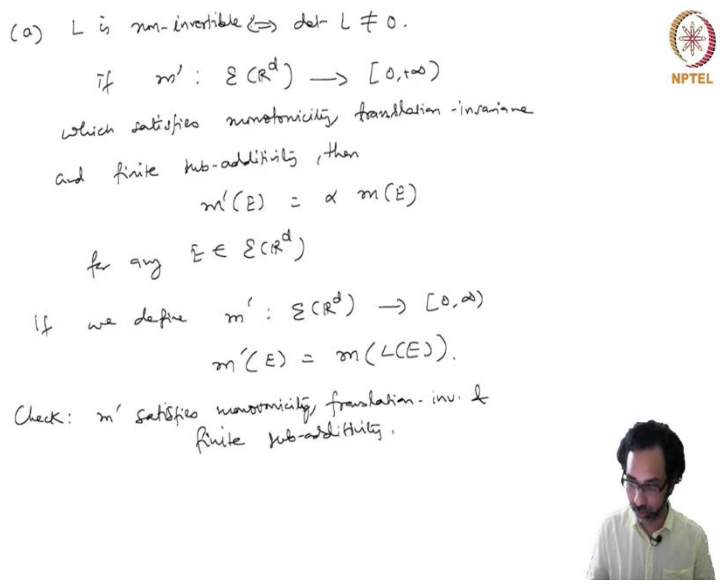


So we come to the case when L is non invertible which means that the determinant of L is not equal to 0. Here I am going to use the fact that if m prime is a map from the elementary subset of Rd to 0 infinity, which satisfies monotonicity translation invariance and finite sub additivity. Then m prime of E is some constant alpha times m of E where E is Jordan measurable for sorry E is elementary for any E in where elementary subset of Rd.

So this was the uniqueness theorem up to scalar multiplication for the Jordan measure. Any other measure which satisfies monotonicity is translation invariance and finite sub additivity must be a constant multiple of the Jordan measure of the elementary measure. So if we define m prime from the elementary subset to 0, infinity as m prime of E is m of L E. Then one can show that this satisfies monotonicity translation invariance and finite sub additivity then check that the m prime satisfies monotonicity translation invariance and finite sub additivity.