Measure Theory

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Module No # 03
Lecture No # 14
Examples of Jordan Measureable sets –II – Part 2

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Now take $S_1 = A_1$ and $S_2 = A_1 \cup C$.

To show: $m(S_2 \setminus S_1) \leq \epsilon \Rightarrow E$ is Jordan measurable.

since $A_1 \subseteq E \subseteq \underbrace{A_1 \cup C}_{\text{check this.}}$

$$m((A_1 \cup C) \setminus A_1) = m(C \setminus A_1)$$
$$\leq m(C) \leq \epsilon.$$

$\Rightarrow E$ is Jordan measurable.

So now take S1 to be A1 and S2 to be A1 union our set C which covered delta E. So I am going to show that the measure of S2 – S1 is this is to show that this is less than or equal to epsilon and this will show that E is Jordan measureable. Because since A1 is the subset of E and E is the subset of A1 union C. So check this this one is trival so check that this ones. So it is just an easy consequence of the way we have been sets A1 and C.

So let us show that this is true so we just write A1 union C – A1 but this is nothing but C – A1 and this is bounded above by the measure of C because of monotonicity property and this is less than or equal to epsilon. So this shows that E is Jordan measureable so this finishes our course proof for the fact that compact convex polytopes in Rd are Jordan measureable.

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2. Regions under graphs of continuous fns:

Let $f : B \to \mathbb{R}$, where B is a closed box in $\mathbb{R}^d$

is a continuous fn.

The set $E = \left\{ (x,t) \in \mathbb{R}^{d+1} : x \in B,\ 0 \le t \le f(x) \right\}$

is Jordan measurable.

To prove that E is Jordan measurable, it suffices to show

that $m^J(\partial E) = 0$

$$\Leftrightarrow m^J\Big(\underbrace{\{(x, f(x)) : x \in B\}}_{\text{Graph of } f \,=\, \partial E} \Big) = 0.$$

] - Check this claim!

Our next class of examples is regions under graphs of continuous functions, so already we have already seen that for hyper planes given by a linear function the region, under the hyper plane is Jordan measureable that corresponded to in the case of compact convex polytope's. Now suppose that f is a continues map from B to R where B is a box in Rd closed box or it suppose this is closed box in Rd and is suppose that this is a continuous function.

So the region under the graph what does it mean this says that the set E let me write this is the set x, t in Rd + 1 so I am raising by 1 dimension higher. Such that x belongs to B and 0 less than equal to t less than or equal to fx. So this set is Jordan measure so to prove this set that E is Jordan measureable it is suffices to show that the outer measure of the topological boundary is 0. So this is the characterization we have just proved and never know that this is equivalent to showing that the topological boundary is given by x fx such that x belongs to B this is 0.

So if we can show that this graph so this is the graph of, f if this graph as Jordan of measure 0 then we are done by the characterization of Jordan measurability in terms of the outer Jordan measurability of the boundary then 0. So I will leave you to check that this is true that graph of, f is indeed delta E, so check this claim now let us show that the outer Jordan measure of this set is 0

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To show: $m^J(\{(x,f(x)) \in \mathbb{R}^{d+1} : x \in B\})$

Grid method: Suppose that $B = I_1 \times \dots \times I_d$

For each $n \in \mathbb{N}$, sub-divide each of the I_k's into sub-intervals $I_{k,j}$, $j = 1, 2, \dots, n+1$ of length $\frac{|I_k|}{n}$. Now suppose that $I_{k,j} = [\xi_{k,j}, \xi_{k,j+1}]$

end-points.

Consider the box

$$\square_{(j_1, j_2, \dots, j_d)} = I_{1,j_1} \times I_{2,j_2} \times \dots \times I_{d,j_d}$$

So to show that the outer Jordan measure of x fx such that so these are point in Rd + 1 such that x is in our box B which was the subset of Rd. Now we use the grid method again so divide so suppose that B is given by the Cartesian product of intervals I1 to Id. Now for each n E N sub divide each of the intervals Ik's into sub intervals let me call them Ikj j = 1, 1 up to n +1 So n+1 sub intervals Ikj the length of Ik over n so this is just I am just rewriting what we did earlier in the case of hyper planes

So we can sub divide each of our intervals into n+1 sub intervals of length the length of Ik over n. So now suppose that Ikj is given by Ckj and Ckj+1 so this is the other n points of our serving tools. Now consider the box denoted by this notation that I used earlier let j, j1 to jd indices line from 1 to n+1 and this is nothing but I1j1, I2j2 up to Idjd so this is a box which divides our box b into smaller pieces when you vary j1, j2, jd or from 1 to n +1. So on this box we will take the supremum of the function fx and the infimum of the function fx.

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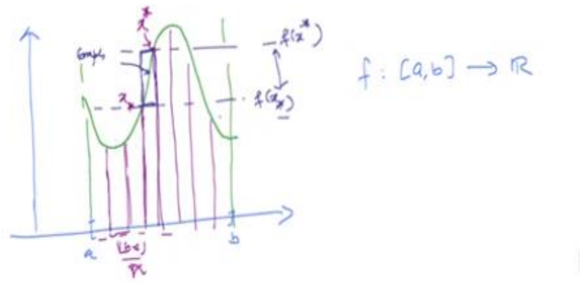
let $x^*_{(j_1,j_2..j_d)} \in \square_{(j_1,j_2,\dots,j_d)}$ be the point

where f attains its supremum value (since f is continuous x^* exists).

Similarly $x_*^{(j_1,j_2..j_d)} \in \square_{(j_1,j_2,\dots,j_d)}$ such that

f attains its inf. value on $\square_{(j_1,j_2..j_d)}$.



So let x star upper star belonging to this box j1, j2, jd be the point where f attains its supremum value this is possible because since f is continuous. So x star exist and similarly x lower star this is the point in our box j1, j2, jd such that f attains its infrimum value on this box. So we have these 2 points x upper star and x lower star that both depends on this indices j1, j2 and jd. So I can actually write j1, j2, jd and here j1, j2, jd so let us see by an example here what we are trying to do here.

So let us suppose that we have f from an interval a, b to R so you have a, and b. And suppose that your function f looks like this so this is a, and this is b and now we are going to subdivide this into various parts. So we are going to use the grid method here only one other coordinates as to be divided so we are dividing this interval a, b to sub intervals each of length utmost 1 over b – a over n. So this length is b - a over n so we will have n intervals and each of them have a length b-a over n.

So now for example for this interval for this one our infrimable we have attain here and our supremum will be attain here. So in this case our x lower star will be this value this one and x upper star will be this value this is and if you see the horizontal line this is f of x lower star and this is f of x upper star. So here then we have a box which covers the graph this is the graph of f and our box will be one side will be given by the sub division of the interval a, b

And other side will be given by the value between fx lower star and fx upper star. So we should just measure the area of all these boxes and this will cover our graph.

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Define

$$E^-_{(j_1,\dots,j_d)} = \left\{(x,t) \in \mathbb{R}^{d+1} \mid x \in \square_{(j_1,\dots,j_d)},\ 0 \le t \le f\left(x_*^{(j_1\dots j_d)}\right)\right\}$$

$$E^+_{(j_1,\dots,j_d)} = \left\{(x,t) \in \mathbb{R}^{d+1} \mid x \in \square_{(j_1,\dots,j_d)},\ 0 \le t \le f\left(x^*_{(j_1\dots j_d)}\right)\right\}$$

The graph of $f(x) = \left\{(x,f(x)) \in \mathbb{R}^{d+1} \mid x \in B\right\}$

is covered by sets $\underline{E^+_{(j_1,\dots,j_d)}} \setminus E^-_{(j_1,\dots,j_d)}$

So in D dimensions we will define E – depending on our indices j1 and jd to be the set xt such that so this is a subset of Rd+1. Such that x belongs to our box j1, jd and 0 less than equal to t less than fx lower star j1, jd. So this is precisely what we did in the k square 1 dimensions so it just takes a while to write down all this things it becomes the bit lengthy in 3 dimensions. But the idea is pretty simple so here we are taking 0 less than equal to t less than fx upper star j1, jd okay.

So now our graphs of fx this is a just a set x, fx belong to Rd + 1 such that x belongs to d. So this set is covered by sets E + j1, jd -E- j1 jd so the outer measure of this graph can be approximated by this sets which are now nice boxes so our measure will be easy to compute.

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$$m^J(\text{Graph of } f) \le m\Big(\bigcup_{(j_1\dots j_d)\in\{1\dots n+1\}} \left(E^+_{(j_1\dots j_d)} \setminus E^-_{(j_1\dots j_d)}\right)\Big)$$

$$\le \sum_{j_1=1}^{n+1}\dots\sum_{j_d=1}^{n+1} \underbrace{\left(m(E^+) - m(E_-)\right)}_{\frac{m(I_1)\dots m(I_d)}{n^d}\left(f(x^*)-f(x_*)\right)}$$

$$= \frac{\prod_{k=1}^{d} m(I_k)}{n^d}\sum_{j_1=1}^{n+1}\dots\sum_{j_d=1}^{n+1}\left(f(x^*)-f(x_*)\right)$$

Because f is cont over a compact set B $\Rightarrow$ f is uniformly cont.

So the measure auto measure of the graph of, f is less than or equal to the measure of the union of this boxes of these E+ j1, j2, jd – E- j1, j2, jd and j1, j2, jd range from 1 to n+1. So now this is a union of elementary sets so this is less than or equal to the sum j1 1 to n + 1 jd 1 to n + 1 measure of E+ I am just dropping the subscripts to make it easier to look at minus the measure of E-. But this is nothing but the measures I1, Id over n to the power d multiplied by fx upper star – fx lower star.

So this we can take outside the summation and we are left with this product of measures mIk k = 1 to d over n to the power d and then we have this sums and fx upper star – fx lower star. So now because f is continuous over a compact set B this implies that f is uniformly continuous.

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So given $\epsilon > 0$, $\exists\, \delta > 0$ s.t.

$$d(x,y) \le \delta \Rightarrow |f(x) - f(y)| \le \epsilon.$$

Choose n large enough s.t.

$$\frac{\sqrt{d}}{n}.c. = \text{diam}\, \square_{(j_1,\dots,j_d)} \le \delta.$$

$$m^*(\text{Graph of } f) \le \frac{\prod_{k=1}^{d} m(I_k)}{n^d} \sum_{j_1=1}^{n+1} \dots \sum_{j_d=1}^{n+1} \underbrace{\left(f(x^*) - f(x_*)\right)}_{\le \epsilon}$$

$$\le \frac{\prod_{k=1}^{d} m(I_k)}{n^d} . \epsilon . (n+1)^d . \quad \text{as } n \to \infty$$

$$\to \left[\prod_{k=1}^{d} m(I_k)\right] . \epsilon .$$

And therefore so given any epsilon greater than 0 where exist at delta such that if the distance between x and y is less than equal to delta then the difference of the modulus absolute value of fx – fy is less than or equal to epsilon. So choose n large enough such that the diameter of this box f any box is than or equal to delta. So because this is given by square root d over n times are constant so this can always be chosen n can be chosen large enough so that it is less than or equal to delta.

And this will imply that after measure after the graph of f and this is less than or equal to the product or the measures Ik to work with d over n to the power d and then we had this sum j1 = 1 to n+ 1 j2 = 1 to, n +1 ad we had this f occurs upper star – f lower star. But if your diameter of this box is less than or equal to delta then this means that this is less than or equal to (()) (20:13). So this means that this is less than equal to the measure the product of this measures over n to the power d times epsilon.

And then we have for each j1, j2 so this should be jd each j1, j2 up to jd simply n+1 terms so this is n+1 to the power d. So this as n goes to infinity this converges to this value mIk time’s epsilon. Now since epsilon is arbitrary we have done because this can be made as small as possible.

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$\therefore \quad m^J(\text{Graph of } f) \leq \alpha \cdot \epsilon. \quad$ for arb. $\epsilon > 0.$

$\Rightarrow \quad m^J(\text{Graph}) = 0.$

$\Rightarrow$ Graph of f is Jordan measurable.

So therefore mJ of the graph after measure is less than or equal to sum constant times epsilon for arbitrary epsilon greater than 0 which means that the outer measure of the graph is 0. So this means that the graph is graph of, f is Jordan measure.