Measure Theory
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Module No # 03
Lecture No # 12
Examples of Jordan Measureable sets -1

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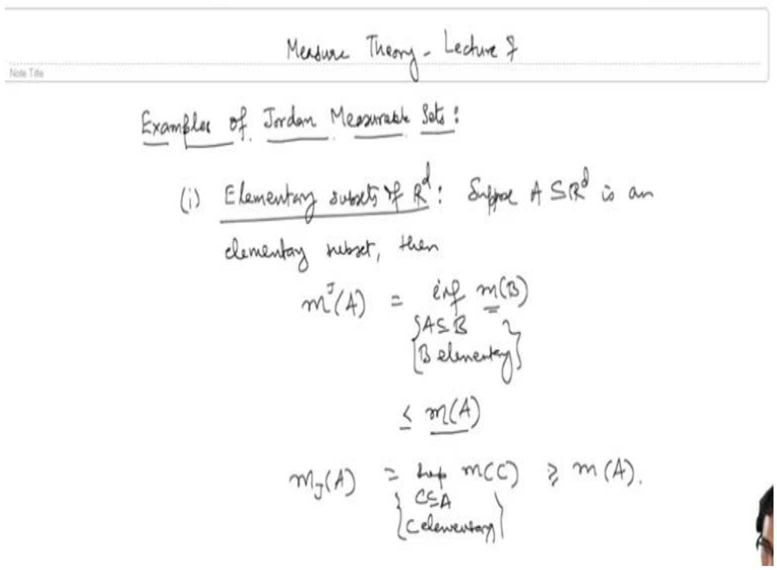

In this lecture we will look at examples of Jordan measureable sets so our first example is simply the elementary subsets of the E clearing space Rd. So these are Jordan measureable so let us see why so suppose that A is an elementary subset of Rd then we can estimate the outer Jordan measure and inner Jordan measure as follows. So the outer Jordan measure is by definition they are supremum sorry infrimum of elementary subsets B which are super sets of A and you take the elementary measure here of B.

Followed that A itself is included in this collection of elementary sets that cover A so therefore this is less than or equal to the elementary measure of A. Now on the other hand the inner Jordan measure is the supremum of elementary sets which are inside A and you take the elementary measure or subsets C which are inside A. Now again in this collection A itself is a member and so therefore this is greater than or equal to the elementary measure of A

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$$m(A) \leq m_J(A) \leq m^J(A) \leq m(A)$$

$$\Rightarrow \quad m_J(A) = m^J(A) = m(A)$$

$\Rightarrow$ A is Jordan measurable with Jordan measure $\underline{m(A)}$ (elementary measure)

(ii) A line segment in $\mathbb{R}^2$ is Jordan measurable.

(Note: A line segment is <u>not</u> elementary in general).

So these 2 are equal is taking together being that the elementary measure is less than or equal to the inner Jordan measure is less than or equal to outer Jordan measure and this is bounded again by the elementary measure of A. Which; implies that the inner Jordan measure and the outer Jordan measure are the same and they are equal to the elementary measure of A. So this means that A is Jordan measureable with Jordan measure given by the elementary measure of A.

So this justifies our notation m of A initially we only defined it if our elementary subsets of Rd and then we denoted m of A as well for the Jordan measure of Jordan measurable subsets of Rd. So since this for any elementary subset which we are now seen is Jordan measureable the Jordan measure is same as the elementary measure. Therefore we can this justifies our use for this notation of this m of A for both elementary measure as well as Jordan measure.

So this means that once we have seen that there are Jordan measureable subsets which are not elementary subset. Then this would mean that our Jordan measure is the strict generalization for the elementary measure and so we could say that we have actually enlarge the glass of subsets of Rd which are which can be given a notion of measures okay. So having completed our proof of elementary measures next we come to the following example.

A line segment in R2 is Jordan measure so this is probably our first example of a Jordan measureable set which is not elementary. So note that a line segment is not elementary in general because even though the each point is elementary a line segment contains uncountable many such points. Therefore you cannot express it as a finite union of elementary sets so let us see why the align segment in R2 is Jordan measureable.

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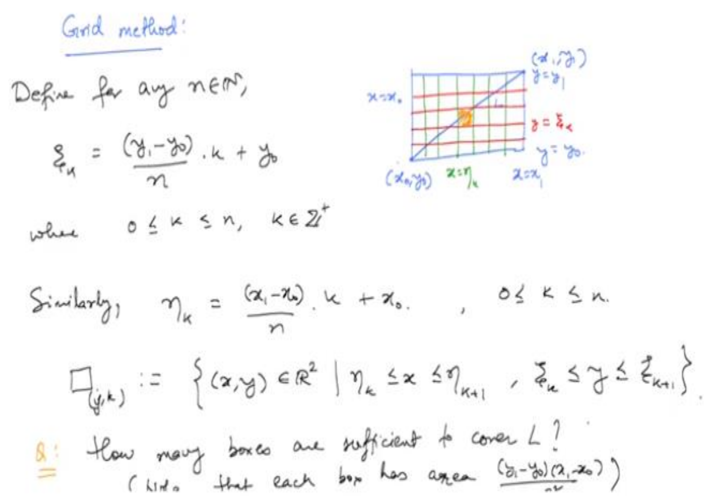

So let me go to another page so here is our line segment let us call it L and suppose that our initial point is x naught, y naught and our final part is x1, y1. So we will (()) (06:08) we will like to call a grid method to determine the Jordan measure for such spaces such sets. So what the grid method says is that you can partition this box so this is the line y = y naught this is the horizontal line is y = y naught of the vertical line is x = x1.

Similarly here the vertical line is x = x naught then the horizontal line is y = y1 so our line segment is contain in this box. Now we can divide our box into smaller boxes so we have this smaller boxes here this is any vertical, any horizontal line is given by y = sum ck where so it is defined for nEm a natural number Ck to be y1 – y0 over n times k + y naught where k ranges between 0 and n. So k here k is a positive integer between 0 and n.

So similarly all of these vertical lines are some x = eta k and we can define eta k to be x1 – x naught over n and xk+ x naught again here k ranges from 0 to 1. So this gives us a decomposition of our big box from x = x naught x = x1 and y = y naught y = y1 into this smaller pieces. So I will let me call this box j, k with index j, k to be the set of points x, y in R2 such that eta k less than equal to x less than equal to eta k + 1 and ck less than equal to y less that equal to ck+1.

So for example this is one of our boxes box j, k given by this partitions eta k and ck's. So now we have to count how many boxes and sufficient to cover this line L. So question is how many boxes are sufficient to cover L and note that each such box has area which is the same as Jordan measure because it is the elementary measure this is given by y1 - y naught x1 – x naught over n or n square.

So each of this box has area equal to this and now if we can count how many boxes are sufficient to cover L then we can have estimate for the outer Jordan measure for L.

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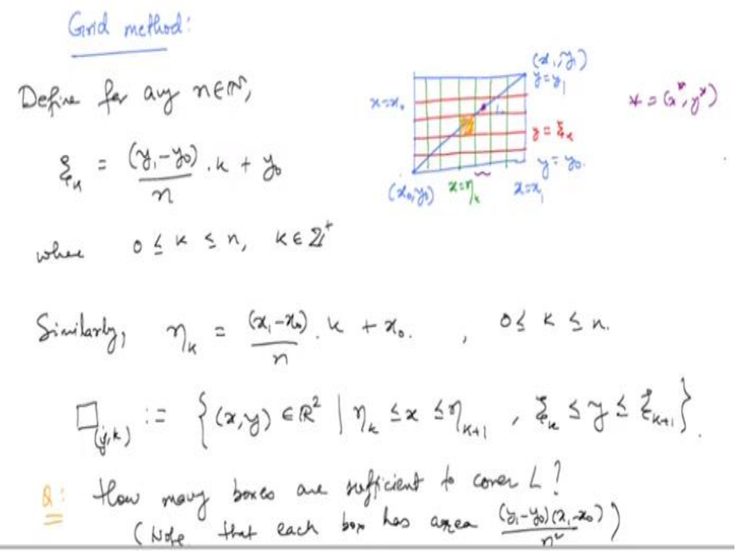

So my claim is that the outer Jordan measure of L = 0 so this is what we would expect because the area of a line is by definition 0. So once we have proved that the outer Jordan measure is 0 by our one of our previous claimer it will show that L is Jordan measureable and in fact Jordan measureable is 0. So we still have to count how many grids squares or grid boxes are required to cover base lines against L.

So note that for any point let us call it x star y star in this line segment we can choose the maximum k such that eta K less than or equal to x star less than or equal to x eta k + 1. So if you so this is basically if we take if we look at this picture let us suppose that our star is our point x star y star then our first we take the indices k for which extra lies in box or strip and then we will try to find out the index in the y direction for which it lies in the horizon position

So once we have been identified vertical strip and horizontal strip then it indices unique box and that will be our box in which x star y star lies.

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Claim: $m^J(L) = 0$

Note that for any point $(x^*, y^*) \in L$,

we can chose the maximum k s.t.

$$\eta_k \le x^* \le \eta_{k+1}$$

$$\eta_k \le x^*$$

$$\iff \frac{(x_1 - x_0)}{n} \cdot k + x_0 \le x^*$$

$$\iff k \le n \cdot \frac{(x^* - x_0)}{(x_1 - x_0)}$$

$$\Rightarrow k = \left\lfloor n \frac{(x^* - x_0)}{(x_1 - x_0)} \right\rfloor$$

So let us do this algebraically so for example if you take eta k less than or equal to k x star. So eta k this is equivalent to saying that x1 – x naught over n times k + x naught equals is less than equal to x star which is the same as saying if the k is less than or equal to n times x star – x naught over x1 – x naught. So the maximum value of k for which this holds is simply the floor function for this value from the right hand side of our inequalities.

So this is the maximum integer less than or equal to this value and now so we have identified our k for which x star lies in the vertical strip.

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Since $y = mx + c$ for the line L,

then $n \cdot \frac{(y^* - y_0)}{(y_1 - y_0)} = n \cdot \frac{(x^* - x_0)}{(x_1 - x_0)}$

(since (x_0, y_0), (x_1, y_1) and (x^*, y^*) are in L).

$\Rightarrow$ $(x^*, y^*) \in \square_{(k,k)}$ for each k in $\{0, 1 .. , n\}$

$\Rightarrow$ We can cover L with $(n+1)$ – boxes

$$L \subseteq \underbrace{\bigcup_{k=0}^{n} \square_{(k,k)}}_{\text{elementary set}}$$

$$m^J(L) \le m\Big(\bigcup_{k=0}^{n} \square_{(k,k)}\Big) \le \sum_{k=0}^{n} \underbrace{m(\square_{(k,k)})}_{(y_1 - y_0)(x_1 - x_0)/n^2} = \sum_{k=0}^{n} \underbrace{\left[\frac{(y_1 - y_0)(x_1 - x_0)}{n^2}\right]} = \frac{(y_1 - y_0)(x_1 - x_0)}{n^2} \cdot (n+1$$

And now we see that since y = mx + c this is just the equation of the line for the line L for some m and c. Then if you take n times y1 – y star – y naught / y1 – y naught this is nothing but n times x star – x naught / x1 – x naught because both our points x star y star x naught y naught as well as x1, y1 all lie in this line y = mx + c. Since; x naught y naught x1, y1 and x star y star are in L.

So therefore we have that this is given by the same formula and so to choose our horizontal strip we see that it is given by the same index k for which we have chosen the horizontal strip or in the for the x coordinate. So this implies that x star y star belongs to the box k, k for each k in 0, 1 up to n. So therefore we see that they maximum of we can cover L with n+1 boxes in our grid. So now we are ready to estimate the outer Jordan measure of L and we see that L is subset of this union k, k box k, k k = 0 to n

So since this is an elementary set so outer Jordan measure is less than or equal to the elementary measure of the union delta k, k k = 0 to, n. And now I can use the finite sub-additivity for the elementary measure to write this as the sum from k = 0 to n and elementary measure for each box and we know that this is nothing but y1 – y naught times x1 – x naught over m square. So therefore this is equal to y1 – y naught times x1 – x naught over m square and then.

So this is a constant in the summation formula so first we can write it as this sum k = 0 to, n of this term. But this term it is not contain n E k so it can be taken out of n summation sign. So this is nothing but y1 – y naught times x1 – x naught over n square times then the number of time the number of indices in the summation.

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$$m^J(L) \leq \underbrace{(y_1-y_0)(x_1-x_0)}_{\text{constant}} \underbrace{\frac{(n+1)}{n^2}} \xrightarrow{n\to\infty} 0$$

Take the limit as $n \to \infty$ $\left(\frac{1}{n}+\frac{1}{n^2}\right) \longrightarrow 0$

$$\Rightarrow \quad m^J(L) = 0$$

$\Rightarrow$ L is Jordan measurable with Jordan measure 0.

So this is nothing but n+1 so in the end we get that the outer Jordan measure is less than or equal to y1 – y naught times x1 – x naught n+1 over n square. So since n is arbitrary you can take the limit as n goes infinity in the limit as n goes to infinity and this gives you that the auto measure auto Jordan measure of L is 0 because this still goes to 0 as n goes to infinity. Because this is a constant and here you have 1 over n + 1 over n square.

So this goes to 0 as n goes to infinity so we see that our outer Jordan measure is 0 therefore L is Jordan measureable with Jordan measure 0. A similar argument once; we can understand what we have done algebraically in this example a similar argument goes on to show that if we have.

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(iii) The region D in $\mathbb{R}^d$ defined by the equation

$$\sum_{j=1}^{d} \mu_j x_j = \alpha \quad , \text{ not all } \mu_j\text{'s are zero. } \alpha \in \mathbb{R},\ \mu_j \in \mathbb{R}.$$

and $a_i \le x_i \le b_i$ for each $i = 1, 2, \ldots d$.

Define for each $n \in \mathbb{N}$,

$$\xi^{(i)}_{k} = \frac{(b_i - a_i)}{n} k + a_i$$

for each i in $\{1, 2, \ldots, d-1\}$, and $k \in \{0, \ldots, n\}$.

Define $\square_{(k_1, k_2, \ldots, k_d)} = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \mid \xi^{(i)}_{k_i} \le x_i \le \xi^{(i)}_{k_i+1}\}$

Q: How many $\square_{(k_1, \ldots, k_d)}$ are sufficient to cover D?

So this is my third example so the region in Rd define by the equation so now this is an equation of hypo plain given by j = 1 to d mu j x j equal to sum constant alpha. So here we have not all Mu j's are 0 and alpha with some constant fixed constant R that was all Mu j's are also R. So here we have equation of hyper plane and again we use the grid method so here sorry so first we have to define so this hyper chromic plane can go and indefinitely so we have to define some finite limits.

So define by the equations and this equation and ai less than or equal to xi less than equal to bi for each i in between 1 and d for the each coordinate you have some fixed bound finite rely interval between ai and bi okay. So we do this same thing as before so we define for each n ck for the ith coordinate which is nothing but bi-ai over n times k + ai okay so each i for each i in 1 to d and k in 0 to, n.

So when k = 0 this is nothing but ai and when k = 1 this is nothing but bi so it ranges between so we have partition each such interval into sub intervals of length 1 over bi – ai over n. So now sorry I am going to do up to d – 1 not d so let me rewrite it so this is that for each i in 1 up to d – 1 and for each i k ranges between 0 and n. So I have chosen here only until d – 1 because once we have decided the (()) (23:31) d -1 then our any point in this hyper plain will lye for exactly 1 index for the dth coordinate for the last coordinate xd.

Let us see this so now we can define hyper a box in our d so this will be dependent upon k1, k2, kd indices. So this will be x1, x2 up to xd ERd such that cki less than or equal to xi less than or equal to cki + 1 Ki. So this is the box between the indices ki and k+1 in the ith coordinate. So now we ask our question again how many such boxes k1, kd are needed are sufficient to cover our region let me call it D here the region cover D okay.

So how many such boxes are required are sufficient to cover this region D and we will use this argument as we did for the line segment in R2.

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Similarly

$$\frac{n\cdot(x_d^*-a_d)}{(b_d-a_d)} \le \frac{\sum_{j=1}^{d-1}(k_j+1)\mu_j(b_j-a_j)}{\sum_{j=1}^{d-1}\mu_j(b_j-a_j)}$$

$$= \left(\frac{\sum_{j=1}^{d-1}k_j\mu_j(b_j-a_j)}{\sum_{j=1}^{d-1}\mu_j(b_j-a_j)}\right) + 1$$

$$\Rightarrow k_d = \left\lfloor \frac{n(x_d^*-a_d)}{(b_d-a_d)} \right\rfloor = \left\lfloor \frac{\sum_{j=1}^{d-1}k_j\mu_j(b_j-a_j)}{\sum_{j=1}^{d-1}\mu_j(b_j-a_j)} \right\rfloor$$

$\Rightarrow$ k_d is determined uniquely by $(k_1, \dots, k_{d-1})$

So let us see how this is done so suppose that you have a point x1 star x2 star xd star in this region D and you can choose k1, k2 up to k d-1 such that x1 let me write it like this such that this point x1 star. So such that ckii less than or equal to xi star less than or equal to cki + 1 i for i in 0 1 up to d-1 and now we have this equation for the hyper plain Mu j x j equal to alpha from j = 1 to b.

So once we have determined d – 1 such indices my claim is that there is only one such index kd for which this point lies in d. So since this equation holds there is only one index kd such that x1 star, x2 star, x3 star belongs to the so this is because we can write now we know that our index kd can be written as the floor of xd star – ad over bd – ad. So this is the same formula as we use for the line segment but we have this expression bd-ad.

This is nothing but the sum of so we can write xd star and ad in terms of coordinates in the lower dimension and so you will have a sum from 1 to d-1 Mu j xj star – aj divided by the same thing here with xj star replace by bj. And now this is greater than or equal to by our choice of k1, k2 up to kd. So this is greater than or equal to because remember that kj is the flow function of n times xj star – aj over bj-aj.

So we can write n times xj star – aj is greater than or equal to Mu j bj –aj so here they should be d -1 kj Mu j times bj-aj j = 1 to b-1 divided by Mu j bj – aj j =1 to d -1 So therefore if we take the slope function of this expression on the right. And similarly we have that this expression is bounded above by kj the sum over kj +1 Mu j bj – aj divided by the same denominator as we had before.

But now notice that this is nothing but j = 1 to d -1 kj Mu j bj –aj over so this is j = 1 to d -1 kj Mu j bj – aj over j = 1 to d-1 Mu j bj – aj + 1 okay. Because there is a +1 here and the denominator and the numerator are same will have +1, which means that the flow function for this quantity d star – ad / bd – ad is the same as the floor of this come here this term written by d-1 kj Mu j bj- aj over Mu j bj – aj j = 1 to d -1.

This implies now that we have of course this term on the left is this kd this implies that kd is determined uniquely by this choice of indices from 1 to k d -1.

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$$\Rightarrow \# \Big\{ \square_{(\underset{(n+1)\text{ choices}}{k_1},\, \underset{(n+1)\text{ choices}}{k_2},\, \ldots,\, \underset{1\text{-choice.}}{k_d})} \text{ sufficient to cover } D \Big\}$$

$$= (n+1)^{d-1}$$

$$m^*(D) \le \sum_{k_1=0}^{n} \sum_{k_2=0}^{n} \cdots \sum_{k_{d-1}=0}^{n} \frac{\left(\prod_{i=1}^{d} (b_i - a_i)\right)}{n^d}$$

$$\le \prod_{i=1}^{d} (b_i - a_i) \cdot \frac{1}{n^d} \cdot (n+1)^{d-1}$$

$$\xrightarrow{n \to \infty} 0. \qquad \Rightarrow m^*(D) = 0.$$

Therefore the number of boxes therefore the number of boxes k1, k2 kd sufficient to cover this region D is nothing but so for each index ki from 1 to d – 1 you will have n+1 choices. So for each one you will have n + 1 choices because k1 for each ki various from 0 to n but for kd you will have just 1 choice. So this is nothing but n + 1 to the power d – 1 so now we can estimate the outer measure for this and this is again nothing but the sum from k1 = 0 to, n k2 = 0 to n and so on kd-1 = 0 to, n.

And here you have in the numerator you will have the volume of the big box which this region is in closed 1 to d over n to the power d. And this is nothing but I = 1 to d bi-ai 1 over n to the power d times n + 1 to the power d-1 and again 1 can check that as n goes to infinity this goes to 0. This implies that the outer measure is 0 and we have shown that d is Jordan measureable with Jordan measure is 0.

So the algebra is the bit tedious once you go to higher dimensions but you can be worked out (()) (34:11) for low dimensional cases. So in many examples solving the lower dimensional case first is very good idea of how to close the higher dimension case where visualization is not possible. So first solve it for 2 or 3 dimensions and then; using the same algebraic techniques you can go to higher dimensions.